

PARABOLIC THEORY OF THE DISCRETE p -LAPLACE OPERATOR

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ABSTRACT. We study the discrete version of the p -Laplacian. Based on its variational properties we discuss some features of the associated parabolic problem. Our approach allows us in turn to obtain interesting information about positivity and comparison principles as well as compatibility with the symmetries of the graph. We conclude briefly discussing the variational properties of a handful of nonlinear generalized Laplacians appearing in different parabolic equations.

1. INTRODUCTION

The *discrete Laplacian* is a well-known object in graph theory. In his seminal investigation on electric circuits [28], Kirchhoff introduced it as

$$\Delta := D - A,$$

where D is the (diagonal) degree matrix and A the adjacency matrix of an undirected graph G . He then went on to observe that for any orientation of the graph the associated (signed) incidence matrix \mathcal{I} satisfies

$$(1.1) \quad \Delta = \mathcal{I}\mathcal{I}^T,$$

that this representation allows us to prove that all the minors of Δ have the same value, and that this value is a natural number that in fact agrees with the number of spanning trees of G ; moreover, 0 is always an eigenvalue whose multiplicity coincides with the number of connected components of G .

A possible reason for considering Δ as a discrete analogous of the usual Laplace operator is that, if one sets up a system of linear equations for the potential f at each node v of an electric circuit in accordance with the (linear) laws of Kirchhoff and Ohm, then

$$\Delta f = 0,$$

i.e., f satisfies a Laplace-type equation like its continuous pendant: this is well explained in [16, Chapt. 1]. Also after Kirchhoff's investigation, Δ has proved a remarkable object in algebraic graph theory. The role of the second smallest eigenvalue of Δ in relation with the study of connectivity properties of a graph has been emphasized since the 1970s by Fiedler and others, cf. the survey article [30]. Moreover, it is nowadays known (see e.g. [34]) that the discrete Laplacian (possibly after a suitable re-normalization) is tightly related to the Dirichlet-to-Neumann (i.e., the voltage-to-current) operator of the metric structure associated with the discrete graph – the so-called *quantum graph*. A thorough theory of linear discrete elliptic operators is nowadays available. This field has experienced successful interactions between the communities of researchers working on graph theory, functional analysis and potential theory, see e.g. [27, 30].

In the 1960s and 1970s, pioneering investigations by (among others) Minty, Rockafellar, and Zemanian aroused broad interest in the theory of nonlinear electric circuits. In those years also the theory of monotone operators and subdifferentials was being substantially developed. It is therefore no surprise that many investigations focused

2010 *Mathematics Subject Classification.* 39A12, 47H20, 05C50.

Key words and phrases. Porous medium equation, Nonlinear semigroups generated by subdifferentials, Operators on discrete graphs, Discrete symmetries.

I would like to thank Daniel Lenz, Robin Nittka, and René Pröpper for fruitful discussions. This research has been supported by the Land Baden-Württemberg in the framework of the *Juniorprofessorenprogramm* – research project on “Symmetry methods in quantum graphs”.

on those nonlinear electric circuits whose associated voltage-to-current operator is maximal monotone. In the same years, the (continuous) p -Laplace operator Δ_p , i.e., the subdifferential of the energy functional

$$(1.2) \quad L^p(\Omega) \ni u \mapsto \frac{1}{p} \|\nabla u\|_{L^p}^p \in [0, \infty]$$

began to be studied by Aronsson, DiBenedetto, Ural'ceva, and many others (cf. [17] for an introduction and survey on this topic). It was in this cultural climate that Yamasaki proposed in [44] the notion of *Dirichlet integral of order $p \in (1, \infty)$* on a graph \mathbf{G} with node set \mathbf{V} : by this the mapping¹

$$\mathcal{E}_p : \mathbb{R}^{\mathbf{V}} \ni f \mapsto \frac{1}{p} \sum_{\substack{\mathbf{v}, \mathbf{w} \in \mathbf{V} \\ \mathbf{v} \sim \mathbf{w}}} |f(\mathbf{v}) - f(\mathbf{w})|^p \in [0, \infty],$$

is meant. It seems that Yamasaki's definition went forgotten until the early 1990s, when \mathcal{E}_p reappeared as part of the final remark of a short note [39] on linear potential theory by Soardi. Ever since, many authors have discussed properties of this energy functional and of its subdifferential, the *discrete p -Laplacian* defined by

$$\Delta_p f(\mathbf{v}) := \sum_{\substack{\mathbf{w} \in \mathbf{V} \\ \mathbf{w} \sim \mathbf{v}}} |f(\mathbf{v}) - f(\mathbf{w})|^{p-2} (f(\mathbf{v}) - f(\mathbf{w})), \quad \mathbf{v} \in \mathbf{V}.$$

(Observe that for $p = 2$ we recover the discrete Laplacian; however the discrete Laplacian is positive definite whereas the common Laplace operator is negative definite).

Most investigations have been devoted to potential theory and eigenvalue problems: we mention, among others, [1, 25, 37] and a long series of articles by Agarwal, O'Regan and coauthors that begins with [26]. In recent years, several works have emphasized the rôle of the discrete p -Laplacian in spectral clustering and image processing, cf. [6, 19, 41, 45].

The introduction of \mathcal{E}_p is motivated by the fact that the finite difference

$$\mathcal{I}^T f(\mathbf{v}, \mathbf{w}) := f(\mathbf{v}) - f(\mathbf{w})$$

can be seen as a discrete analogue of (minus) a directional derivative – this fact is classically used in numerical analysis, giving rise to the notion of *backward difference operator*. If we impose an orientation on a graph \mathbf{G} and regard functions from \mathbf{V} to \mathbb{R} as *difference 0-forms* (or *0-chains*) and functions from \mathbf{E} to \mathbb{R} as *difference 1-forms* (or *1-chains*), the corresponding derivation is exactly $\mathcal{I}^T : \mathbb{R}^{\mathbf{V}} \rightarrow \mathbb{R}^{\mathbf{E}}$, the transpose of the oriented $|\mathbf{V}| \times |\mathbf{E}|$ incidence matrix

$$(1.3) \quad \mathcal{I} := \mathcal{I}^+ - \mathcal{I}^-,$$

where $\mathcal{I}^+ := (\iota_{\mathbf{ve}}^+)$ and $\mathcal{I}^- := (\iota_{\mathbf{ve}}^-)$ are defined by

$$\iota_{\mathbf{ve}}^+ := \begin{cases} 1 & \text{if } \mathbf{v} \text{ is initial endpoint of } \mathbf{e}, \\ 0 & \text{otherwise,} \end{cases} \quad \iota_{\mathbf{ve}}^- := \begin{cases} 1 & \text{if } \mathbf{v} \text{ is terminal endpoint of } \mathbf{e}, \\ 0 & \text{otherwise,} \end{cases}$$

respectively. Hence, Yamasaki's functional \mathcal{E}_p can be written as

$$\mathcal{E}_p := \mathbb{R}^{\mathbf{V}} \ni f \mapsto \frac{1}{p} \|\mathcal{I}^T f\|_{\ell^p(\mathbf{E})}^p \in [0, \infty],$$

and

$$(1.4) \quad \Delta_p : \mathbb{R}^{\mathbf{V}} \ni f \mapsto \mathcal{I}(|\mathcal{I}^T f|^{p-2} \mathcal{I}^T f) \in \mathbb{R}^{\mathbf{V}},$$

in apparent analogy with the continuous p -Laplacian operator and its associated energy functional.

Just like in the continuous case, where (after applying the divergence theorem) the conservation law

$$\frac{d}{dt} \int_{\Omega} \phi(t, x) dx = - \int_{\Omega} \operatorname{div} j(t, x) dx + \int_{\Omega} f(t, x) dx$$

¹ Throughout this paper we write $\mathbf{v} \sim \mathbf{w}$ if there is an edge between \mathbf{v} and \mathbf{w} .

is considered, in the discrete case we have the conservation law

$$\frac{d}{dt} \sum_{v \in V} \phi(t, v) = - \sum_{e \in E} (\mathcal{I}j)(t, e) + \sum_{v \in V} f(t, v)$$

and Fick's law

$$j(t, e) = -c(\mathcal{I}^T \phi)(t, e)$$

for a general function c can be used to derive a general (time-continuous, space-discrete) differential equation governing a flow on G , cf. [22, § 2.5.5]. Indeed, for $c \equiv 1$ we obtain exactly the linear heat equation for the discrete Laplacian. Considering different terms c we obtain different equations. In particular, opting for the nonlinear Darcy's law

$$c = c(t, e, (\mathcal{I}^T \phi)(t, e)) := |\mathcal{I}^T \phi(t, e)|^{p-2},$$

we end up with

$$\frac{d}{dt} \sum_{v \in V} \phi(t, v) = - \sum_{e \in E} \mathcal{I}(|\mathcal{I}^T \phi|^{p-2} \mathcal{I}^T \phi)(t, v) + \sum_{v \in V} f(t, v, \phi(t, v)).$$

In view of (1.4), this is the *summed* version of the parabolic problem associated with the discrete p -Laplacian Δ_p – let us call it the *discrete p -heat equation*. Observe that, unlike its continuous counterpart, it is a *backward* evolution equation.

The present paper is devoted to study the basic features of this equation and is structured as follows. After an introductory section where the relevant functional setting is introduced, we proceed by formulating in Section 3 our main well-posedness result for the discrete p -heat equation. A Galerkin-type method can be applied to yield a solution and, unlike in the usual context of PDEs on domains, we can even interpret the converging Galerkin sequence yielded as a sequence of solutions to the same equation on induced subgraphs (i.e., on growing subsystems). Using the theory of nonlinear Dirichlet forms, we can show that these solutions are associated with semigroup of nonlinear contraction on all ℓ^q -spaces, $1 \leq q \leq \infty$, as well as on c_0 . This is done by discussing the invariance property of several relevant closed convex subsets under the semigroup generated by $-\mathcal{E}_p$. We show that the nonlinear semigroup generated by the discrete p -Laplacian consists of irreducible, sub-Markovian operators. In the special case of $p = 2$, these results had already been obtained in [27].

Similar methods are applied in Section 4 to prove how these semigroups interact with the symmetries of the graph. This is more delicate than in the continuous case, since the discrete p -Laplacian is not a local operator.

Finally, in the Appendix we overview a few popular generalizations of the discrete Laplacian, propose some more and suggest how the variational structure of all can be used to discuss a broad class of discrete nonlinear diffusion-type problems, including a discretized porous medium equation.

Because the discrete p -heat equation is simply a dynamical system, most of our results follow from the standard theory of ordinary differential equations whenever G is a finite graph. The actually interesting case is hence that of infinite graphs. We refer to [15, Chapter 8] for an introduction to infinite graph theory. On the analytical side, standard references for nonlinear Cauchy problems on Banach spaces are the monographs [4, 38].

2. GENERAL SETTING AND THE ENERGY FUNCTIONAL

We consider throughout a (finite or countable) set V and a (non-symmetric) relation

$$E \subset \{(v, w) \in V \times V : v \neq w\}$$

such that for any two elements $v, w \in V$ at most one of the pairs $(v, w), (w, v)$ belongs to E . We refer to the elements of V and E as *nodes* and *edges*, respectively, and regard G as a graph whose incidence matrix is \mathcal{I} introduced in (1.3). If $e = (v, w)$, we denote the initial and terminal endpoint of e by

$$e_0 := v \quad \text{and} \quad e_1 := w, \quad \text{respectively, where } e = (v, w).$$

Inspired by [27, § 1], throughout this article we consider G as a *weighted graph*, i.e., as a quadruple (V, E, μ, ν) where $\mu : E \rightarrow \mathbb{R}_+$ and $\nu : V \rightarrow \mathbb{R}_+$ (the unweighted case corresponds to $\mu \equiv 1$ and $\nu \equiv 1$).

Remark 2.1. *Actually, by construction \mathbf{G} is oriented. This is reflected already in the definition of the (signed) incidence matrix \mathcal{I} . However, orientation is only a technical tool that will make some parametrizations easier: Unless otherwise underlined, throughout most of this paper only the absolute value of the numbers*

$$\mathcal{I}^T f(e) = \sum_{v \in V} \iota_{ve} f(v) = f(e_0) - f(e_1)$$

will play a rôle. Thus, all our results (with the exception of those in Section 4) are still valid if any of the edges is given the opposite orientation.

We have already introduced the (possibly infinite) incidence matrix

$$\mathcal{I} : \mathbb{R}^E \rightarrow \mathbb{R}^V \quad \text{and its transposed} \quad \mathcal{I}^T : \mathbb{R}^V \rightarrow \mathbb{R}^E.$$

Along with them, the structure of the *undirected* graph associated with \mathbf{G} is mirrored by the $|V| \times |V|$ *adjacency matrix* $\mathcal{A} := (\alpha_{vw})$ defined by

$$(2.1) \quad \alpha_{vw} := \begin{cases} \mu(e) & \text{if } e = (v, w) \text{ or } e = (w, v) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2. *A graph $\mathbf{G} := (V, E, \mu, \nu)$ is called outward locally finite if*

$$\deg^+(v) := \sum_{e \in E} \iota_{ve}^+ \mu(e) \leq M_v^+ \quad \text{for all } v \in V \text{ and some } M_v^+ > 0.$$

It is called inward locally finite if

$$\deg^-(v) := \sum_{e \in E} \iota_{ve}^- \mu(e) \leq M_v^- \quad \text{for all } v \in V \text{ and some } M_v^- > 0.$$

It is called locally finite if it is both inward and outward locally finite, i.e., if

$$\deg(v) := \deg^+(v) + \deg^-(v) \leq M_v \quad \text{for all } v \in V \text{ and some } M_v > 0.$$

If $\deg^+ \in O(\nu)$, i.e., if there exists $M^+ > 0$ s.t.

$$\deg^+(v) \leq M^+ \nu(v) \quad \text{for all } v \in V,$$

then \mathbf{G} is called outward uniformly locally finite. We define inward uniform local finiteness and uniform local finiteness likewise.

We say that \mathbf{G} has finite surface and finite volume if

$$|V|_\nu := \|\nu\|_{\ell^1(V)} < \infty \quad \text{and} \quad |E|_\mu := \|\mu\|_{\ell^1(E)} < \infty,$$

respectively.

Observe that a graph with finite volume is necessarily uniformly locally finite.

Assumptions 2.3. *Throughout this paper we impose the following assumptions on $\mathbf{G} := (V, E, \mu, \nu)$.*

- $\mu(e) > 0$ for all $e \in E$.
- $\nu(v) > 0$ for all $v \in V$.
- \mathbf{G} is locally finite.

The weights μ, ν define in a natural way a metric and measure structure on \mathbf{G} . As we will see in Lemma 3.5, the (discrete) p -Laplace–Beltrami operator associated with these coefficients is given by

$$(2.2) \quad \Delta_p f := \frac{1}{\nu} \mathcal{I}(\mu |\mathcal{I}^T f|^{p-2} \mathcal{I}^T f), \quad f \in \mathbb{R}^V.$$

However, we would like to allow for a general class of (non-degenerate) elliptic operators, possibly in non-divergence form. To this aim, we will have to consider general weights a and d . In order to avoid degeneracies, certain compatibility conditions should hold.

Assumptions 2.4. Throughout this paper the functions $a : \mathbf{E} \rightarrow \mathbb{R}$ and $d : \mathbf{V} \rightarrow \mathbb{R}$ are assumed to satisfy the following conditions.

- There exist $\kappa, K > 0$ s.t.

$$\kappa a(\mathbf{e}) \leq \mu(\mathbf{e}) \leq K a(\mathbf{e}) \quad \text{for all } \mathbf{e} \in \mathbf{E}.$$

- There exist $\theta, \Theta > 0$ s.t.

$$\theta d(\mathbf{v}) \leq \nu(\mathbf{v}) \leq \Theta d(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{V}.$$

Clearly, if there exist $\zeta, Z > 0$ s.t.

$$\zeta \leq \mu \leq Z \quad (\text{resp., s.t. } \zeta \leq \nu \leq Z),$$

then one can choose $a \equiv 1$ (resp., $d \equiv 1$).

For $p \in [1, \infty]$ we will consider the weighted sequence spaces $\ell_a^p(\mathbf{E})$ and $\ell_d^q(\mathbf{V})$ defined by

$$\|u\|_{\ell_a^p}^p := \sum_{\mathbf{e} \in \mathbf{E}} |u(\mathbf{e})|^p a(\mathbf{e}) \quad \text{and} \quad \|f\|_{\ell_d^q}^q := \sum_{\mathbf{v} \in \mathbf{V}} |f(\mathbf{v})|^q d(\mathbf{v}),$$

along with

$$w_{a,d}^{1,p,2}(\mathbf{V}) := \{f \in \ell_d^2(\mathbf{V}) : \mathcal{I}^T f \in \ell_a^p(\mathbf{E})\}.$$

In other words,

$$w_{a,d}^{1,p,2}(\mathbf{V}) = \left\{ f \in \ell_d^2(\mathbf{V}) : \|\mathcal{I}^T f\|_{\ell_a^p}^p = \sum_{\mathbf{e} \in \mathbf{E}} a(\mathbf{e}) |f(\mathbf{e}_0) - f(\mathbf{e}_1)|^p < +\infty \right\}.$$

Remark 2.5. Let $p \in [1, \infty]$. Observe that another way of expressing uniform local finiteness of \mathbf{G} , i.e., that $\deg \in O(\nu)$, is saying that $\ell_{\deg}^p(\mathbf{V})$ is continuously embedded in $\ell_\nu^p(\mathbf{V})$. Also observe that under the Assumptions 2.4 the weighted norms $\|\cdot\|_{\ell_\mu^p}$ and $\|\cdot\|_{\ell_a^p}$, and also $\|\cdot\|_{\ell_\nu^p}$ and $\|\cdot\|_{\ell_d^p}$ are equivalent. In particular, $w_{\mu,\nu}^{1,p,2}$ and $w_{a,d}^{1,p,2}$ agree.

In the theory of discrete calculus, the rule of thumb is to perform the following substitutions:

- scalar functions \rightarrow vectors of the node space,
- vector fields \rightarrow vectors of the edge space,
- gradient of a scalar function ϕ at a point $x \rightarrow$ evaluation of $\mathcal{I}^T \phi$ at an edge \mathbf{e} ,
- divergence of a vector field ψ at a point $x \rightarrow$ evaluation of $\mathcal{I} \psi$ at a node \mathbf{v} .

This interplay between continuous and discrete setting is evocative and already thoroughly investigated (see e.g. [22]). Actually, $w_{a,d}^{1,p,2}(\mathbf{V})$ will play the rôle of a weighted Sobolev space.

Lemma 2.6. For all $p, q \in [1, \infty)$, $\ell_a^q(\mathbf{V})$ and $\ell_d^p(\mathbf{E})$ are separable Banach spaces, and so is $w_{a,d}^{1,p,2}(\mathbf{V})$ with respect to the norm defined by

$$\|f\|_{w_{a,d}^{1,p,2}} := \sqrt{\|f\|_{\ell_d^2}^2 + \|\mathcal{I}^T f\|_{\ell_a^p}^p}.$$

For all $p \in [1, \infty]$, $w_{a,d}^{1,p,2}(\mathbf{V})$ is continuously and densely embedded into $\ell_d^2(\mathbf{V})$. If moreover $p \in [1, \infty)$, then $w_{a,d}^{1,p,2}(\mathbf{V})$ is separable. If $p \in (1, \infty)$, then $w_{a,d}^{1,p,2}(\mathbf{V})$ is uniformly convex (and hence reflexive).

Proof. Separability of $\ell_d^p(\mathbf{V})$ and $\ell_a^q(\mathbf{E})$ follows from countability of \mathbf{V} , hence of \mathbf{E} . Considering

$$w_{a,d}^{1,p,2}(\mathbf{V}) \ni f \mapsto (f, \mathcal{I}^T f) \in \ell_d^2(\mathbf{V}) \times \ell_a^p(\mathbf{E}),$$

which is an isometry if the Cartesian product on the right is endowed with the ℓ^2 -norm, shows that $w_{a,d}^{1,p,2}$ is separable for all $p \in [1, \infty)$, and uniformly convex if $p \in (1, \infty)$. Furthermore, since the space $c_{00}(\mathbf{V}) \subset \mathbb{R}^{\mathbf{V}}$ of functions with finite support is dense in $\ell_d^2(\mathbf{V})$, so is $w_{a,d}^{1,p,2}(\mathbf{V})$. \square

Lemma 2.7. *Under our standing Assumptions 2.3, let $p \in [1, \infty]$. If G is outward uniformly (resp., inward uniformly, uniformly) locally finite, then \mathcal{I}^+ (resp., $\mathcal{I}^-, \mathcal{I}$) is bounded from $\ell_a^p(\mathbf{E})$ to $\ell_d^p(\mathbf{V})$. The converse implication holds for $p \in [1, \infty)$. It also holds for $p = \infty$ if additionally there exist $\tilde{\mu}, \tilde{\nu} > 0$ s.t. $\mu(\mathbf{e}) \leq \tilde{\mu}$ and $\tilde{\nu} < \nu(\mathbf{v})$ for all $\mathbf{e} \in \mathbf{E}$ and $\mathbf{v} \in \mathbf{V}$.*

In particular, G is uniformly locally finite if and only if $w_{a,d}^{1,p,2}(\mathbf{V}) = \ell_d^2(\mathbf{V})$ for all $p \in [2, \infty)$.

This result is a generalization of [7, Prop. 6] to weighted graphs.

Proof. Take $f : \mathbf{V} \rightarrow \mathbb{R}$ and observe that

$$\sum_{\mathbf{v} \in \mathbf{V}} \iota_{\mathbf{ve}}^+ |f(\mathbf{v})|^p = |f(\mathbf{e}_0)|^p = \max_{\mathbf{v} \in \mathbf{V}} \iota_{\mathbf{ve}}^+ |f(\mathbf{v})|^p \quad \text{for all } \mathbf{e} \in \mathbf{E} \text{ and all } p \in [1, \infty),$$

because there exists exactly one $\mathbf{v} \in \mathbf{V}$ s.t. $\iota_{\mathbf{ve}}^+ \neq 0$.

Take first $p \in [1, \infty)$. By Fubini's theorem, one has for all $f : \mathbf{E} \rightarrow \mathbb{R}$

$$\begin{aligned} \|\mathcal{I}^{+T} f\|_{\ell_\mu^p}^p &= \sum_{\mathbf{e} \in \mathbf{E}} |f(\mathbf{e}_0)|^p \mu(\mathbf{e}) \\ &= \sum_{\mathbf{e} \in \mathbf{E}} \left(\sum_{\mathbf{v} \in \mathbf{V}} |f(\mathbf{v})|^p \iota_{\mathbf{ve}}^+ \right) \mu(\mathbf{e}) \\ &= \sum_{\mathbf{v} \in \mathbf{V}} |f(\mathbf{v})|^p \sum_{\mathbf{e} \in \mathbf{E}} \iota_{\mathbf{ve}}^+ \mu(\mathbf{e}) \\ &= \sum_{\mathbf{v} \in \mathbf{V}} |f(\mathbf{v})|^p \deg^+(\mathbf{v}) = \|f\|_{\ell_{\deg}^p}^p. \end{aligned}$$

This shows that \mathcal{I}^{+T} is an isometry from $\ell_\mu^p(\mathbf{E})$ to $\ell_{\deg}^p(\mathbf{V})$, hence by Remark 2.5 it is bounded from $\ell_\mu^p(\mathbf{E})$ to $\ell_\nu^p(\mathbf{V})$ if and only if G is uniformly locally finite. One concludes that \mathcal{I}^{+T} is bounded from $\ell_a^p(\mathbf{E})$ to $\ell_d^p(\mathbf{V})$ if and only if G is uniformly locally finite.

For $p = \infty$ one has

$$\begin{aligned} \|\mathcal{I}^{+T} f\|_{\ell_\mu^\infty} &= \max_{\mathbf{e} \in \mathbf{E}} |f(\mathbf{e}_0)| \mu(\mathbf{e}) \\ &= \max_{\mathbf{e} \in \mathbf{E}} \max_{\mathbf{v} \in \mathbf{V}} \iota_{\mathbf{ve}}^+ |f(\mathbf{v})| \mu(\mathbf{e}) \\ &\leq \sum_{\mathbf{e} \in \mathbf{E}} \max_{\mathbf{v} \in \mathbf{V}} \iota_{\mathbf{ve}}^+ |f(\mathbf{v})| \mu(\mathbf{e}) \\ &= \max_{\mathbf{v} \in \mathbf{V}} |f(\mathbf{v})| \sum_{\mathbf{e} \in \mathbf{E}} \iota_{\mathbf{ve}}^+ \mu(\mathbf{e}) \\ &= \max_{\mathbf{v} \in \mathbf{V}} |f(\mathbf{v})| \deg(\mathbf{v}) = \|f\|_{\ell_{\deg}^\infty}. \end{aligned}$$

To prove the converse implication, take a sequence $(\mathbf{v}_n)_{n \in \mathbb{N}} \subset \mathbf{V}$ s.t.

$$n\nu(\mathbf{v}_n) \leq \deg^+(\mathbf{v}_n)$$

and consider the functions $u_n : \mathbf{E} \rightarrow \mathbb{R}$ defined for all $n \in \mathbb{N}$ by

$$u_n(\mathbf{e}) := \mathbb{1}_{\mathbf{E}_n^+},$$

where $E_n^+ := \{e \in E : \iota_{v_n e}^+ \neq 0\}$, the set of edges outgoing from v_n . Then $\|u_n\|_{\ell_\mu^\infty} \leq \tilde{\mu}$ for all $n \in \mathbb{N}$, but

$$\begin{aligned} \|\mathcal{I}^+ u_n\|_{\ell_\nu^\infty} &= \max_{v \in V} \left| \sum_{e \in E} \iota_{ve}^+ u_n(e) \right| \nu(v) \\ &\geq \sum_{e \in E_n^+} (\iota_{v_n e}^+ \mu(e)) \frac{\tilde{\nu}}{\tilde{\mu}} \\ &= \deg^+(v_n) \frac{\tilde{\nu}}{\tilde{\mu}} \\ &\geq n \frac{\tilde{\nu}^2}{\tilde{\mu}}, \end{aligned}$$

hence $\lim_{n \rightarrow \infty} \|\mathcal{I}^+ u_n\|_{\ell_\nu^\infty} = +\infty$. This concludes the proof. \square

Remark 2.8. (1) Recall that the oriented incidence matrix \mathcal{I} of a finite graph G and κ connected components has rank $|V| - \kappa$, see e.g. [21, § 8.3.1]. Hence, \mathcal{I} is never surjective (resp., \mathcal{I}^T is never injective). On the other hand, \mathcal{I} is injective (resp., \mathcal{I}^T is surjective) if and only if it has rank $|E|$, which is the case exactly when G is a forest. It is easy to see that, more generally, \mathcal{I}^T is surjective whenever G is a uniformly locally finite forest.

(2) Let now G be infinite and connected. If $f \in \mathbb{R}^V$, $f \not\equiv 0$, satisfies $\mathcal{I}^T f = 0$, then necessarily f is constant and hence does not belong to $\ell_d^p(V)$ unless $\|1\|_{\ell_d^p} < \infty$, i.e., unless G has finite surface (for $p < \infty$) or unless ν is bounded (for $p = \infty$).

Taking into account Remark 2.8, a direct computation yields the following.

Corollary 2.9. Let $p \in [1, \infty)$. Then a necessary condition for the operator $\mathcal{I}^T : \ell_d^p(V) \rightarrow \ell_a^p(E)$ to be an isomorphism is that G be infinite; a sufficient one is that G be a uniformly locally finite tree with infinite surface.

Following [44], we consider an energy functional²

$$\mathcal{E}_p : f \mapsto \frac{1}{p} \|\mathcal{I}^T f\|_{\ell_a^p}^p = \frac{1}{p} \sum_{e \in E} a(e) |f(e_0) - f(e_1)|^p, \quad f \in \mathbb{R}^V.$$

The definition of \mathcal{E}_p is clearly independent of the orientation of G . While by allowing the value $\mathcal{E}_p(f) = +\infty$ we can (and indeed shall) consider \mathcal{E}_p as defined on the whole space $\ell_d^2(V)$, one also considers the *effective domain*

$$D(\mathcal{E}_p) := \{f \in \ell_d^2(V) : \mathcal{E}_p(f) < +\infty\},$$

which is clearly non-empty: one says that \mathcal{E}_p is *proper*. In fact, $D(\mathcal{E}_p) = w_{a,d}^{1,p,2}(V)$. In view of the mentioned analogies between discrete and continuous calculus \mathcal{E}_p is the discrete companion to the functional introduced in (1.2).

Remark 2.10. A natural analogue of our setting arises in the theory of electric circuits, following the scheme

- $u(v) \rightarrow$ voltage at v ,
- $\mathcal{I}^T u(e) \rightarrow$ potential difference between the endpoints of e ,
- $a(e) \rightarrow$ conductance of the branch corresponding to e ,
- $a(e)\mathcal{I}^T u(e) \rightarrow$ current flowing between the endpoints of e ,
- setting Dirichlet boundary conditions on a certain set V_0 of nodes \rightarrow grounding the nodes in V_0 ,
- replacing the values of a function u in two nodes $v, w \in V$ by their average \rightarrow shorting the nodes of the corresponding electric circuit,
- replacing the value of a in an edge e by 0 \rightarrow cutting the branch of the corresponding electric circuit.

² For the sake of notational simplicity, we stipulate that $\psi(e)$ will be written as (indifferently) either $\psi(v, w)$ or $\psi(w, v)$, whenever e is an edge with endpoints v, w and ψ is a function from E to \mathbb{R} .

Observe for example that cutting branches induces a shift in the Rayleigh-type quotient

$$\frac{\mathcal{E}_p(f)}{\|f\|_{\ell_d^p}^p},$$

which is the natural object to minimize if one looks for eigenvalues, cf. [6] (i.e., for those $\lambda \in \mathbb{R}$ s.t.

$$\lambda|\varphi(\mathbf{v})|^{p-2}\varphi(\mathbf{v}) = \mathcal{L}_p\varphi(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V},$$

has a non-trivial solution φ). This shift is in accordance with the intuition that diffusion is slower on a graph with fewer edges and corresponds to Lord Rayleigh's Monotonicity Law ([16, § 1.4]). We will not go into details of this interplay and refer instead to the beautiful booklet [16], where the linear theory is explained in detail.

For $a \equiv 1$, \mathcal{E}_1 is the energy associated with the discrete mean curvature operator discussed e.g. in [45], while \mathcal{E}_2 agrees with the quadratic form associated with (1.1), and in fact the subdifferential of \mathcal{E}_2 is just the discrete Laplacian. If \mathbf{G} is uniformly locally finite, then by Lemma 2.7 \mathcal{I} and hence Δ_2 are bounded linear operators. Accordingly, the initial value problem associated with

$$\dot{\varphi}(t, \mathbf{v}) + \Delta_2\varphi(t, \mathbf{v}) = 0, \quad t \in \mathbb{R}, \mathbf{v} \in \mathbf{V}.$$

is well-posed in $\ell_d^2(\mathbf{V})$ and the exponential matrices $e^{t\Delta_2}$, $t \in \mathbb{R}$, yield its solutions. Fiedler has showed that the largest eigenvalue of Δ_2 is larger than the maximal node degree of \mathbf{G} . Thus, dropping the assumption of uniform local finiteness Δ_2 becomes an unbounded operator whose spectrum is not contained in any left half-plane, hence not the generator of a strongly continuous semigroup on $\ell_d^2(\mathbf{V})$. Still, it can be proved by methods based on quadratic forms that $-\Delta_2$ is a generator. Our aim in the next section is to show that, just like in the special case of $p = 2$, most properties of the continuous p -Laplacian carry over to its discrete counterpart, despite the latter being non-local.

3. WELL-POSEDNESS RESULTS

In view of the analogy between discrete and continuous calculus, the correct discrete pendant of the continuous p -Laplacian is the nonlinear operator on $\mathbb{R}^{\mathbf{V}}$ defined by

$$(3.1) \quad \Delta_p f := \mathcal{I}(|\mathcal{I}^T f|^{p-2}\mathcal{I}^T f), \quad f \in \mathbb{R}^{\mathbf{V}},$$

in the case of an unweighted graph. More generally, we are going to consider the nonlinear operator given by

$$(3.2) \quad \mathcal{L}_p f := \frac{1}{d}\mathcal{I}(a|\mathcal{I}^T f|^{p-2}(\mathcal{I}^T f)), \quad f \in \mathbb{R}^{\mathbf{V}}.$$

Definition 3.1. Let \mathbf{G} be a graph. A family of graphs $(\mathbf{G}_n)_{n \in \mathbb{N}}$ is called growing if for all $n, m \in \mathbb{N}$ with $n \leq m$ \mathbf{G}_n is an induced subgraph of \mathbf{G}_m . It is said to exhaust \mathbf{G} if for all $n \in \mathbb{N}$

- \mathbf{G}_n is an induced subgraph of \mathbf{G} , i.e., $\mathbf{G}_n = (\mathbf{V}_n, \mathbf{E}_n, \mu_n, \nu_n)$ with $\mu_n = \mu|_{\mathbf{E}_n}$ and $\nu_n = \nu|_{\mathbf{V}_n}$; and
- $\bigcup_{n \in \mathbb{N}} \mathbf{V}_n = \mathbf{V}$.

The following is the main result of this section.

Theorem 3.2. Let $T > 0$, $f_0 \in w_{a,d}^{1,p,2}(\mathbf{V})$ and $f \in L^2(0, T; \ell_d^2(\mathbf{V}))$. Then the Cauchy problem

$$(HEp) \quad \begin{cases} \dot{\phi}(t, \mathbf{v}) &= -\mathcal{L}_p\phi(t, \mathbf{v}) + f(t), & t \in [0, T], \mathbf{v} \in \mathbf{V}, \\ \phi(0, \mathbf{v}) &= f_0(\mathbf{v}), & \mathbf{v} \in \mathbf{V}, \end{cases}$$

admits a unique solution $\phi \in H^1(0, T; \ell_d^2(\mathbf{V})) \cap L^\infty(0, T; w_{a,d}^{1,p,2}(\mathbf{V}))$.

Furthermore, there is a growing family of graphs $(G_n)_{n \in \mathbb{N}}$ that exhausts G and such that the sequence of solutions $(\phi_n)_{n \in \mathbb{N}}$ to the Cauchy problem³

$$(HEp^{(n)}) \quad \begin{cases} \dot{\phi}_n(t, v) &= -\mathcal{L}_p \phi_n(t, v) + f(t), & t \in [0, T], v \in V_n, \\ \phi_n(t, v) &= 0, & t \in [0, T], v \in N(V_n), \\ \phi_n(0, v) &= f_0(v), & v \in V_n, \end{cases}$$

converges to ϕ , weakly in $H^1(0, \infty; \ell_d^2(V))$ and weakly* in $L^\infty(0, \infty; w_{a,d}^{1,p,2}(V))$.

If moreover $f \equiv 0$ and $f_0 \in \ell_d^q(V)$ for some $q \in [1, \infty]$ (resp., $f_0 \in c_{0d}(V)$), then $\phi \in C(\mathbb{R}_+; \ell_d^q(V))$ (resp., $\phi \in C(\mathbb{R}_+; c_{0d}(V))$) and $(\phi_n)_{n \in \mathbb{N}}$ converges to ϕ also weakly* in $L^\infty(\mathbb{R}_+; \ell_d^q(V))$ (resp., in $L^\infty(\mathbb{R}_+; c_{0d}(V))$).

Thus, we can approximate a dynamical system on the whole G by a sequence of dynamical systems on induced subgraphs G_n . Before proving it, we will need a number of results.

Lemma 3.3. *Let $p \in (1, \infty)$. Then the proper functional \mathcal{E}_p is convex and $\mathcal{E}_p + \frac{\omega}{2} \|\cdot\|_{\ell_d^2}^2$ is coercive for any $\omega > 0$. Furthermore, \mathcal{E}_p is continuously (Fréchet) differentiable as a functional on $w_{a,d}^{1,p,2}(V)$ and lower semicontinuous as a functional on $\ell_d^p(V)$.*

Proof. Since for all $p \in (1, \infty)$

$$(3.3) \quad \mathcal{E}_p = \frac{1}{p} \|\cdot\|_{\ell_a^p(E)}^p \circ \mathcal{I}^T \quad \text{on } w_{a,d}^{1,p,2}(V),$$

the composition of a convex and a linear mapping, the functional \mathcal{E}_p is convex – hence so is any perturbation by another convex functional, and in particular $\mathcal{E}_p(\cdot) + \frac{\omega}{2} \|\cdot\|_{\ell_d^2}^2$ is convex and coercive for any $\omega > 0$.

In view of (3.3), and because \mathcal{I}^T is a bounded linear operator from $w_{a,d}^{1,p,2}(V)$ to $\ell_a^p(E)$, in order to check continuous differentiability of \mathcal{E}_p it suffices to observe that the functional $\|\cdot\|_{\ell_a^p(E)}^p$ on $\ell_a^p(E)$ is continuously differentiable for $p \in (1, \infty)$. Thus, \mathcal{E}_p is in particular lower semicontinuous as a functional on $w_{a,d}^{1,p,2}(V)$, and lower semicontinuity as a functional on $\ell_d^2(V)$ follows from [38, Lemma IV.5.2]. \square

Remark 3.4. *One can see that if additionally G is uniformly locally finite, then \mathcal{E}_p is lower semicontinuous also for $p = 1$. However, we cannot invoke again [38, Lemma IV.5.2], as this result relies upon reflexivity of the effective domain. Nevertheless, by Lemma 2.7 \mathcal{I}^T is a bounded linear operator from $\ell_d^2(V)$ to $\ell_a^2(E)$. Moreover, lower semicontinuity of $\|\cdot\|_{\ell_a^1(E)}$ (viewed as a function from $\ell_a^2(E)$ to $[0, +\infty]$) follows from Fatou's Lemma. In view of*

$$\mathcal{E}_1 = \|\cdot\|_{\ell_a^1(E)} \circ \mathcal{I}^T : \ell_d^2(V) \rightarrow [0, +\infty],$$

we also deduce lower semicontinuity of \mathcal{E}_1 . From now on, we will always restrict to the case of $p > 1$.

Lemma 3.3 yields that \mathcal{E}_p has a subdifferential, which is at most single-valued by Lemma 2.6.

Lemma 3.5. *Let $p \in (1, \infty)$. Under the Assumptions 2.4, the subdifferential of \mathcal{E}_p with respect to $\ell_d^2(V)$ is given by*

$$\mathcal{L}_p f(v) := \frac{1}{d(v)} \sum_{\substack{w \in V \\ w \sim v}} a(v, w) |f(v) - f(w)|^{p-2} (f(v) - f(w)), \quad v \in V,$$

with maximal domain.

³ Here and in the following $N(W)$ denote the neighborhood – in the graph theoretical sense – of the subgraph of G induced by some node subset $W \subset V$: i.e., $N(W)$ is the set of all $v \in V$ such that $\mu(v, w) \neq 0$ for at least one $w \in W$.

Proof. Let $f \in \ell_d^2(\mathbf{V})$. If $p \in (1, \infty)$, by Lemma 2.6 and because \mathcal{E}_p is Fréchet differentiable whenever restricted to $w_{a,d}^{1,2,p}(\mathbf{V})$, its subdifferential $\partial \mathcal{E}_p f \subset \ell_d^2(\mathbf{V})$ is either empty or a singleton. By virtue of [32, Lemma 2.8.8], for f in the domain of \mathcal{L}_p the only element of \mathcal{L}_p is given by

$$\begin{aligned} \partial \mathcal{E}_p f = g &\Leftrightarrow \mathcal{E}'_p(f)\phi = (g|\phi)_{\ell_d^2} \quad \forall \phi \in \ell_d^2(\mathbf{V}) \\ &\Leftrightarrow \sum_{\mathbf{e} \in \mathbf{E}} a(\mathbf{e}) |(\mathcal{I}^T f)(\mathbf{e})|^{p-2} (\mathcal{I}^T f)(\mathbf{e}) (\mathcal{I}^T \phi)(\mathbf{e}) = \sum_{\mathbf{v} \in \mathbf{V}} g(\mathbf{v}) \phi(\mathbf{v}) d(\mathbf{v}) \quad \forall \phi \in \ell_d^2(\mathbf{V}) \\ &\Leftrightarrow \frac{1}{d(\mathbf{v})} \mathcal{I} (a |\mathcal{I}^T f|^{p-2} (\mathcal{I}^T f)) (\mathbf{v}) = g(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}, \end{aligned}$$

in accordance with (3.2) (recall that by assumption $d(\mathbf{v}) > 0$ for all $\mathbf{v} \in \mathbf{V}$). More explicitly, for all $\mathbf{v} \in \mathbf{V}$

$$\begin{aligned} \partial \mathcal{E}_p f(\mathbf{v}) &= \frac{1}{d(\mathbf{v})} \sum_{\mathbf{e} \in \mathbf{E}} \iota_{\mathbf{v}\mathbf{e}} \left(a(\mathbf{e}) \left| \sum_{\mathbf{w} \in \mathbf{V}} \iota_{\mathbf{w}\mathbf{e}} f(\mathbf{w}) \right|^{p-2} \left(\sum_{\mathbf{w} \in \mathbf{V}} \iota_{\mathbf{w}\mathbf{e}} f(\mathbf{w}) \right) \right) \\ &= \frac{1}{d(\mathbf{v})} \sum_{\mathbf{e} \in \mathbf{E}} \iota_{\mathbf{v}\mathbf{e}} \left(a(\mathbf{e}) |f(\mathbf{e}_0) - f(\mathbf{e}_1)|^{p-2} (f(\mathbf{e}_0) - f(\mathbf{e}_1)) \right) \\ &= \frac{1}{d(\mathbf{v})} \sum_{\substack{\mathbf{w} \in \mathbf{V} \\ \mathbf{w} \sim \mathbf{v}}} a(\mathbf{v}, \mathbf{w}) |f(\mathbf{w}) - f(\mathbf{v})|^{p-2} (f(\mathbf{w}) - f(\mathbf{v})). \end{aligned}$$

This completes the proof. \square

Proposition 3.6. *Let $p \in (1, \infty)$. The set of eigenvectors of \mathcal{L}_p for the eigenvalue 0 is a subspace of $\ell_d^2(\mathbf{V})$ spanned by the characteristic vectors of the connected component of \mathbf{G} with finite surface, in the sense of Definition 2.2.*

Proof. If $f \in \ell_d^2(\mathbf{V})$ is an eigenfunction of \mathcal{L}_p with eigenvalue 0, then $\mathcal{I}^T f = 0$. By Remark 2.8 this implies $f = 0$ if and only if each connected component of \mathbf{G} has infinite surface. \square

By Lemmas 3.3 and 3.5, a direct application of [38, Prop. IV.5.2] yields the following.

Proposition 3.7. *Let $p \in (1, \infty)$ and let a, d satisfy Assumptions 2.4. Let $T > 0$, $f_0 \in w_{a,d}^{1,p,2}(\mathbf{V})$ and $f \in L^2(0, T; \ell_d^2(\mathbf{V}))$. Then the Cauchy problem (HEp) admits a unique solution $\phi \in H^1(0, T; \ell_d^2(\mathbf{V})) \cap L^\infty(0, T; w_{a,d}^{1,p,2}(\mathbf{V}))$.*

If $f = 0$, then the solution is given by a strongly continuous semigroup of nonlinear contractions and $\phi(t, \cdot) \in D(\mathcal{L}_p)$ for all $t > 0$.

In order to prove relevant properties of the p -heat equation one often wants to prove that certain relevant subsets of the state space are left invariant by $(e^{-t\mathcal{L}_p})_{t \geq 0}$. This can be done by a nonlinear generalization of the Beurling–Deny criteria due to Barthélemy [2]. In the continuous case most proofs are made easy by the locality of the operator. In the discrete case locality fails to hold: The key point in our proofs will then be that the considered orthogonal projections map vectors in $\ell_d^2(\mathbf{V})$ into vectors with smaller oscillation.

We will need the following apparent property of the ℓ^p -norm.

Lemma 3.8. *Define a function*

$$f_{k,p} : [0, \infty) \ni \alpha \mapsto |k + \alpha|^p + |k - \alpha|^p \in [0, \infty).$$

Then $f_{k,p}$ is strongly monotone increasing for all $k \in \mathbb{R}$ and $p \in [1, \infty)$.

The following is a discrete analogue of various results proved or summarized in [11, § 4.1]. In the special case of $p = 2$, these results have been obtained in [27] already.

Proposition 3.9. *Let $p \in (1, \infty)$. The strongly continuous nonlinear semigroup $(e^{-t\mathcal{L}_p})_{t \geq 0}$ generated by $-\mathcal{L}_p$ is (sub)Markovian, in the sense of [11, Def. 2.2], i.e., it is order preserving,*

$$f \leq g \quad \Rightarrow \quad e^{-t\mathcal{L}_p} f \leq e^{-t\mathcal{L}_p} g \quad \forall t \geq 0,$$

and $\|\cdot\|_{\ell_d^\infty}$ -contractive,

$$\|e^{-t\mathcal{L}_p} f - e^{-t\mathcal{L}_p} g\|_{\ell_d^\infty} \leq \|f - g\|_{\ell_d^\infty} \quad \forall t \geq 0.$$

Moreover, it is positivity preserving,

$$f \geq 0 \quad \Rightarrow \quad e^{-t\mathcal{L}_p} f \geq 0 \quad \forall t \geq 0.$$

Proof. Because \mathcal{E}_p is nonnegative, the condition

$$(3.4) \quad \mathcal{E}_p(f \wedge g) + \mathcal{E}_p(f \vee g) \leq \mathcal{E}_p(f) + \mathcal{E}_p(g) \quad \text{for all } f, g \in w_{a,d}^{1,p,2}(\mathbf{V}),$$

is sufficient (and necessary) for the semigroup to be order preserving, cf. [2, Cor. 2.2]. In particular, the semigroup is order preserving if for all $\mathbf{e} \in \mathbf{E}$ and all $f, g \in w_{a,d}^{1,p,2}(\mathbf{V})$

$$|\mathcal{I}^T(f \wedge g)(\mathbf{e})|^p + |\mathcal{I}^T(f \vee g)(\mathbf{e})|^p \leq |\mathcal{I}^T f(\mathbf{e})|^p + |\mathcal{I}^T g(\mathbf{e})|^p.$$

This can be proved taking $\mathbf{e} \in \mathbf{E}$ and $f, g \in w_{a,d}^{1,p,2}(\mathbf{V})$ and dividing the four possible cases

- $f(\mathbf{e}_0) \leq g(\mathbf{e}_0)$ and $f(\mathbf{e}_1) \leq g(\mathbf{e}_1)$,
- $g(\mathbf{e}_0) \leq f(\mathbf{e}_0)$ and $g(\mathbf{e}_1) \leq f(\mathbf{e}_1)$,
- $g(\mathbf{e}_0) \leq f(\mathbf{e}_0)$ and $f(\mathbf{e}_1) \leq g(\mathbf{e}_1)$,
- $f(\mathbf{e}_0) \leq g(\mathbf{e}_0)$ and $g(\mathbf{e}_1) \leq f(\mathbf{e}_1)$.

The assertion clearly holds in the first two cases. In order to check its validity in the third and fourth case, we prove that for all $x, y, w, z \in \mathbb{R}$ with $x \geq y$ and $w \geq z$

$$|y - z|^p + |x - w|^p \leq |x - z|^p + |y - w|^p,$$

i.e.,

$$|k + \alpha|^p + |k - \alpha|^p \geq |k + \beta|^p + |k - \beta|^p,$$

where

$$k := \frac{x + y - w - z}{2}, \quad \alpha := \frac{x - y + w - z}{2}, \quad \beta := \frac{-x + y + w - z}{2}.$$

In fact, if $x \geq y$ and $w \geq z$, then

$$y - x + w - z \leq y - x - w + z \leq x - y - w + z,$$

i.e., $|\alpha| > |\beta|$ and the assertion follows from Lemma 3.8. The other cases can be treated likewise.

Let us now prove that the semigroup is $\|\cdot\|_{\ell_d^\infty}$ -contractive. First of all, because \mathcal{E}_p is homogeneous (of degree p), by [11, Cor. 3.7] we have to prove that

$$\begin{aligned} & \mathcal{E}_p \left(\frac{g + (f - g + 1)_+}{2} + \frac{g - (f - g - 1)_-}{2} \right) \\ & + \mathcal{E}_p \left(\frac{f - (f - g + 1)_+}{2} + \frac{f + (f - g - 1)_-}{2} \right) \leq \mathcal{E}_p(f) + \mathcal{E}_p(g) \quad \text{for all } f, g \in w_{a,d}^{1,p,2}(\mathbf{V}). \end{aligned}$$

As above, it suffices to take one $\mathbf{e} \in \mathbf{E}$ and $f, g \in w_{a,d}^{1,p,2}(\mathbf{V})$ and to check that

$$\begin{aligned} & \left| \mathcal{I}^T \frac{g + (f - g + 1)_+}{2}(\mathbf{e}) + \mathcal{I}^T \frac{g - (f - g - 1)_-}{2}(\mathbf{e}) \right|^p \\ & + \left| \mathcal{I}^T \frac{f - (f - g + 1)_+}{2}(\mathbf{e}) + \mathcal{I}^T \frac{f + (f - g - 1)_-}{2}(\mathbf{e}) \right|^p \\ & \leq |\mathcal{I}^T f(\mathbf{e})|^p + |\mathcal{I}^T g(\mathbf{e})|^p \quad \text{for all } f, g \in w_{a,d}^{1,p,2}(\mathbf{V}). \end{aligned}$$

or equivalently

$$\begin{aligned} & \left| \frac{y + (x - y + 1)_+}{2} - \frac{z + (w - z + 1)_+}{2} + \frac{y - (x - y - 1)_-}{2} - \frac{z - (w - z - 1)_-}{2} \right|^p \\ & + \left| \frac{x - (x - y + 1)_+}{2} - \frac{w - (w - z + 1)_+}{2} + \frac{x + (x - y - 1)_-}{2} - \frac{w + (w - z - 1)_-}{2}(\mathbf{e}_1) \right|^p \\ & \leq |x - w|^p + |y - z|^p \end{aligned}$$

for all $x, y, w, z \in \mathbb{R}$. Proving this inequality in the nine possible cases

- $|x - y| \leq 1$ and $|w - z| \leq 1$,
- $|x - y| \leq 1$ and $w - z \geq 1$,
- $|x - y| \leq 1$ and $w - z \leq -1$,
- $x - y \leq -1$ and $|w - z| \leq 1$,
- $x - y \leq -1$ and $w - z \geq 1$,
- $x - y \leq -1$ and $w - z \leq -1$,
- $x - y \geq 1$ and $|w - z| \leq 1$,
- $x - y \geq 1$ and $w - z \geq 1$,
- $x - y \geq 1$ and $w - z \leq -1$,

is tedious but not difficult. E.g., in the second case (the first being trivial) one has to check that

$$\left| x - \frac{w - z + 1}{2} \right|^p + \left| y - \frac{w + z - 1}{2} \right|^p \leq |x - w|^p + |y - z|^p.$$

Also in this case, this condition can be re-written as

$$|k + \alpha|^p + |k - \alpha|^p \geq |k + \gamma|^p + |k - \gamma|^p,$$

where again

$$k := \frac{x + y - w - z}{2}, \quad \alpha := \frac{-x + y + w - z}{2},$$

and

$$\gamma := \frac{x - y - 1}{2}.$$

Under the assumption that $(|x - y| \leq 1 \text{ and } w - z \geq 1)$, one has

$$x - y - w + z \leq x - y - 1 \leq y - x + w - z,$$

and hence $|\gamma| < |\alpha|$, whence the assertion follows, by Lemma 3.8.

Finally, by [9, Cor. 15.5] the semigroup is positivity preserving if and only if

$$\mathcal{E}_p(f^+) \leq \mathcal{E}_p(f) \quad \text{for all } f \in \ell_d^2(\mathbf{V}).$$

This is actually satisfied, because for all $\mathbf{e} \in \mathbf{E}$ dividing the cases

- $f(\mathbf{e}_0), f(\mathbf{e}_1) \geq 0$,
- $f(\mathbf{e}_0), f(\mathbf{e}_1) \leq 0$,
- $f(\mathbf{e}_0) \leq 0 \leq f(\mathbf{e}_1)$,
- $f(\mathbf{e}_0) \geq 0 \geq f(\mathbf{e}_1)$.

one sees that

$$|f^+(\mathbf{e}_0) - f^+(\mathbf{e}_1)| \leq |f(\mathbf{e}_0) - f(\mathbf{e}_1)|.$$

This yields the claim. □

Remark 3.10. *In fact we have just showed more than we have stated: our proof yields that the same assertions hold for any restriction of \mathcal{L}_p , in particular for any restriction to induced subgraphs.*

Remark 3.11. Let \mathcal{E} be a coercive, convex, lower semicontinuous functional on a Hilbert space H . By [4, Prop. 4.5], if a closed convex subset of H is left invariant under $(e^{-t\partial\mathcal{E}})_{t \geq 0}$ for all $t \geq 0$, then the same holds under all resolvent operators $J_\lambda := (\text{Id} + \lambda\partial\mathcal{E})^{-1}$ for all $\lambda \in (0, \infty)$.

1) In this way one easily sees that $e^{-t\mathcal{L}_p}$ is not homogeneous (of any degree) for any $p \in (1, \infty)$ and any $t > 0$, although \mathcal{L}_p is homogeneous of degree p . However, $e^{-t\mathcal{L}_p}0$ because $\mathcal{E}_p(0) = 0$.

(2) In particular, Proposition 3.9 yields the following comparison result:

Let $\lambda > 0$. If $f \geq 0$, then the solution φ of the elliptic problem

$$(3.5) \quad \lambda\varphi(v) + \mathcal{L}_p\varphi(v) = f(v), \quad v \in V,$$

satisfies $\varphi \geq 0$. Moreover, if φ_1, φ_2 are the solutions of the elliptic equation with inhomogeneous data f_1, f_2 and if $f_1 \geq f_2$, then also $\varphi_1 \geq \varphi_2$. This result is comparable to results that are known to hold on finite unweighted graphs, cf. [26].

Lemma 3.12. Let $p \in (1, \infty)$. The strongly continuous nonlinear semigroup $(e^{-t\mathcal{L}_p})_{t \geq 0}$ extrapolates to a family of nonlinear semigroups on $\ell_d^q(V)$ for all $q \in [1, \infty]$ (strongly continuous if $q \neq \infty$) as well as on

$$c_{0d}(V) := \{f : V \rightarrow \mathbb{R} : \lim_{n \rightarrow \infty} \int f(v_n) d(v_n) = 0\}.$$

Formally speaking, this result yields well-posedness of a dynamical system only if one is able to determine the domain of \mathcal{L}_p . In the linear case ($p = 2$) this has been done in [27].

Proof. By Proposition 3.9, contractivity of $(e^{-t\mathcal{L}_p})_{t \geq 0}$ with respect to the norm of $\ell_d^\infty(V)$ yields that the semigroup on $\ell_d^2(V)$ extends to a contractive semigroup on the closure of $\ell_d^2(V)$ in the $\ell_d^\infty(V)$ -norm, i.e., in $c_{0d}(V)$. By duality we obtain a contractive semigroup on $\ell_d^1(V)$, and finally again by duality a contractive semigroup on $\ell_d^\infty(V)$. Now, the assertion follows applying the non-linear Riesz–Thorin-type interpolation theorem due to Browder, see [11, Thm. 3.6] for a version tailored for our setting. \square

Proof of Theorem 3.2. As already emphasized in Remark 3.10, the proof of Proposition 3.9 shows the $\|\cdot\|_{\ell_d^\infty}$ -contractivity of the semigroup generated by the subdifferential of each restriction of \mathcal{E}_p . Consider the sequence of finite dimensional subspaces of $\ell_d^2(V)$ of the form

$$\ell_d^2(V_n) \equiv \{f : \ell_d^2(V) : f(v_m) = 0 \ \forall m > n\} \equiv \mathbb{R}^{V_n},$$

where $V_n := \{v_1, \dots, v_n\}$. Accordingly, the restriction of \mathcal{E}_p to $\ell_d^2(V_n)$ is the functional given by

$$\mathcal{E}_p^{(n)}(f)(v) := \frac{1}{p} \sum_{\substack{v, w \in V_n \\ v \sim w}} a(v, w) |f(v) - f(w)|^p + \frac{1}{p} \sum_{\substack{v \in V_n \\ w \in N(V_n)}} a(v, w) |f(v)|^p, \quad v \in V.$$

A direct computation yields that $-\mathcal{L}_p^{(n)} := \partial\mathcal{E}_p^{(n)}$ is the restriction of $-\mathcal{L}_p$ to the space of function on the subgraph induced by $V_n \cup N(V_n)$, with Dirichlet conditions on $N(V_n)$. Hence, the associated Cauchy problem is $(\text{HEp}^{(n)})$. Fitting to the present setting the usual arguments that yield on convergence of the Galerkin scheme (e.g., following the proof of [9, Thm. 6.1, Part 3]), one sees that the sequence of solutions of $(\text{HEp}^{(n)})$ converges (up to subsequences) to the solution found in Proposition 3.7 – weakly in $H^1(0, T; \ell_d^2(V))$ and weakly* in $L^\infty(0, T; w_{a,d}^{1,p,2}(V))$.

Let now $f \equiv 0$. Then for all $n \in \mathbb{N}$ the solution to each $(\text{HEp}^{(n)})$ is given by a strongly continuous semigroup of nonlinear contractions, and the same holds for the the solution to (HEp) . Because the restricted energy functional $\mathcal{E}_p^{(n)}$ is a proper, convex and lower semicontinuous functional, (minus) its subdifferential generates a semigroup on $\ell_d^2(V_n)$, which is $\|\cdot\|_{\ell_d^\infty}$ -contractive by Proposition 3.9. By the same interpolation argument used in the proof of Lemma 3.12, this semigroup extrapolates to $c_{0d}(V)$ as well as to $\ell_d^q(V)$ for all $q \in [1, \infty]$. In particular, for all $f_0 \in \ell_d^q(V)$

$$\|e^{-\cdot\mathcal{L}_p^{(n)}} f_0\|_{L^\infty(0, \infty; \ell_d^q(V_n))} \leq \|f_0|_{V_n}\|_{\ell_d^q(V_n)} \leq \|f_0\|_{\ell_d^q(V)},$$

due to the fact that $\mathcal{E}_p(0) = 0$ and hence $e^{-t\mathcal{L}_p^{(n)}}0 = 0$, and because of the contractivity of the semigroup. Because for all $q \in (1, \infty]$ $L^\infty(0, \infty; \ell_d^q(\mathbf{V})) = L^1(0, \infty; \ell_d^{q'}(\mathbf{V}))'$ is the dual of a separable Banach space, by the theorem of Banach–Alaoglu for all $t > 0$ the bounded sequence $(e^{-t\mathcal{L}_p^{(n)}}f_0)_{n \in \mathbb{N}} \subset L^\infty(0, \infty; \ell_d^q(\mathbf{V}))$ has a weak*-convergent subsequence. \square

Recall that a semigroup on a Banach space X is called *irreducible* if the only ideals of X that are left invariant under it are the trivial ones.

Proposition 3.13. *Let $p \in (1, \infty)$. Then the graph \mathbf{G} is connected if and only if the semigroup $(e^{-t\mathcal{L}_p})_{t \geq 0}$ is irreducible.*

In the special case $p = 2$, by [33, Thm. 2.9 and Def. 2.8] connectedness hence implies that if $f \geq 0$ but $f \neq 0$, then $e^{-t\Delta_2}f > 0$: this property of the discrete Laplacian is already known, see e.g. [27, Cor. 2.9].

The proof of sufficiency is due to René Pröpper (Ulm).

Proof. All ideals of $\ell_d^2(\mathbf{V})$ are of the form $\ell_d^2(\mathbf{V}_0) \equiv \{f \in \ell_d^2(\mathbf{V}) : \text{supp } f \subset \mathbf{V}_0\}$ for some subset $\mathbf{V}_0 \subset \mathbf{V}$, and the associated orthogonal projections are given by the restriction operators $P_{\mathbf{V}_0} := \mathbb{1}_{\mathbf{V}_0}$.

Now, assume $\ell_d^2(\mathbf{V}_0)$ to be invariant under $(e^{-t\Delta_p})$, or equivalently that

$$(3.6) \quad \mathcal{E}_p(P_{\mathbf{V}_0}f) \leq \mathcal{E}_p(f) \quad \text{for all } f \in \ell_d^2(\mathbf{V}).$$

We have to show that \mathbf{V}_0 is a trivial subset of \mathbf{V} , i.e., $\mathbf{V}_0 = \mathbf{V}$ or $\mathbf{V}_0 = \emptyset$.

In fact, if $\mathbf{V}_0 \neq \mathbf{V} \neq \mathbf{V}_0^C$ there are two adjacent nodes $\mathbf{v}_0 \in \mathbf{V}_0$ and $\mathbf{v}_1 \in \mathbf{V}_0^C$. Set

$$\tilde{\mathbf{V}} := \mathbf{V} \setminus \{\mathbf{v}_0, \mathbf{v}_1\},$$

so that \mathbf{E} is partitioned into $(\mathbf{v}_0, \mathbf{v}_1), \mathbf{E}^0, \mathbf{E}^1, \tilde{\mathbf{E}}$, where for $i \in \{0, 1\}$ \mathbf{E}^i consist of those edges *other than* $(\mathbf{v}_0, \mathbf{v}_1)$ one of whose endpoints is \mathbf{v}_i (regardless of their orientation), and $\tilde{\mathbf{E}} := \mathbf{E} \setminus ((\mathbf{v}_0, \mathbf{v}_1) \cup \mathbf{E}^0 \cup \mathbf{E}^1)$. In other words,

$$\begin{aligned} p\mathcal{E}_p(g) &= a(\mathbf{v}_0, \mathbf{v}_1)|g(\mathbf{v}_0) - g(\mathbf{v}_1)|^p + \sum_{\substack{\mathbf{w} \sim \mathbf{v}_0 \\ \mathbf{v} \neq \mathbf{v}_1}} a(\mathbf{v}_0, \mathbf{w})|g(\mathbf{v}_0) - g(\mathbf{w})|^p \\ &\quad + \sum_{\substack{\mathbf{w} \sim \mathbf{v}_1 \\ \mathbf{v} \neq \mathbf{v}_0}} a(\mathbf{v}_1, \mathbf{w})|g(\mathbf{v}_1) - g(\mathbf{w})|^p + \sum_{\mathbf{e} \in \tilde{\mathbf{E}}} a(\mathbf{e})|g(\mathbf{e}_0) - g(\mathbf{e}_1)|^p \quad \text{for all } g \in w_{a,d}^{1,p,2}(\mathbf{V}). \end{aligned}$$

Let now $f \in \ell_d^2(\mathbf{V})$ be defined by

$$f(\mathbf{v}) := \begin{cases} x, & \text{if } \mathbf{v} = \mathbf{v}_0, \\ 1, & \text{if } \mathbf{v} = \mathbf{v}_1, \\ 0, & \text{otherwise,} \end{cases}$$

for some $x > 0$ to be determined later. Accordingly,

$$P_{\mathbf{V}_0}f(\mathbf{v}) := \begin{cases} x, & \text{if } \mathbf{v} = \mathbf{v}_0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$p\mathcal{E}_p(f) = a(\mathbf{v}_0, \mathbf{v}_1)|x - 1|^p + \sum_{\substack{\mathbf{w} \sim \mathbf{v}_0 \\ \mathbf{v} \neq \mathbf{v}_1}} a(\mathbf{v}_0, \mathbf{w})|x|^p + \sum_{\substack{\mathbf{w} \sim \mathbf{v}_1 \\ \mathbf{v} \neq \mathbf{v}_0}} a(\mathbf{v}_1, \mathbf{w}).$$

(observe that the sums on the RHS are finite, because they are less than $\deg(\mathbf{v}_0)$ and $\deg(\mathbf{v}_1)$, respectively – recall that \mathbf{G} is assumed to be locally finite) whilst

$$p\mathcal{E}_p(P_{\mathbf{V}_0}f) = a(\mathbf{v}_0, \mathbf{v}_1)|x|^p + \sum_{\substack{\mathbf{w} \sim \mathbf{v}_0 \\ \mathbf{v} \neq \mathbf{v}_1}} a(\mathbf{v}_0, \mathbf{w})|x|^p.$$

Accordingly,

$$p\mathcal{E}_p(f) - p\mathcal{E}_p(P_{V_0}f) = a(v_0, v_1) (|x-1|^p - |x|^p) + \sum_{\substack{w \sim v_1 \\ v \neq v_0}} a(v_1, w) < 0,$$

choosing x large enough, thereby contradicting (3.6). This implies that indeed $V_0 = \emptyset$ or $V_0 = G$.

Conversely, each subspace of $\ell_d^2(V)$ consisting of functions over a connected component is apparently left invariant under $(e^{-t\mathcal{L}_p})_{t \geq 0}$. \square

4. SYMMETRIES

Graphs bearing some symmetry are well-studied objects of graph theory. In this section we are going to show how non-trivial symmetries of (HEp) arise if the underlying graph enjoys special symmetry properties. Because \mathcal{E}_p and \mathcal{L}_p do not depend on the orientation of G , we can fortunately neglect it throughout this section.

To begin with, we need a simple lemma.

Lemma 4.1. *Let H_1, H_2 be two Hilbert spaces and $E_1 : H_1 \rightarrow [0, \infty]$ and $E_2 : H_2 \rightarrow [0, \infty]$ two proper, lower semicontinuous, convex functions. Let Σ be a bounded linear operator from H_1 to H_2 . Then Σ intertwines with the semigroup generated by $-\partial E_1$ and $-\partial E_2$, i.e.,*

$$e^{-t\partial E_1}\Sigma = \Sigma e^{-t\partial E_2} \quad \text{for all } t \geq 0$$

if and only if

$$E_1(Lf + \Sigma^*Rg) + E_2(\Sigma Lf + g - Rg) \leq E_1(f) + E_2(g) \quad \text{for all } f \in H_1, g \in H_2,$$

where

$$L := (I_{H_1} + \Sigma^*\Sigma)^{-1} \quad \text{and} \quad R := (I_{H_2} + \Sigma\Sigma^*)^{-1}.$$

Proof. One checks directly that Σ intertwines with the semigroups if and only if the graph of Σ , i.e., the closed subspace

$$\text{Graph}(\Sigma) := \left\{ \begin{pmatrix} x \\ \Sigma x \end{pmatrix} \in H_1 \times H_2 \right\}$$

is invariant under the matrix semigroup

$$e^{-t\partial \mathbf{E}} := \begin{pmatrix} e^{-t\partial E_1} & 0 \\ 0 & e^{-t\partial E_2} \end{pmatrix}, \quad t \geq 0,$$

where $\mathbf{E} := E_1 \oplus E_2$. A classical formula due to von Neumann yields that the orthogonal projection of $H_1 \times H_2$ onto $\text{Graph}(\Sigma)$ is given by

$$P_{\text{Graph}(\Sigma)} = \begin{pmatrix} L & \Sigma^*R \\ \Sigma L & I_{H_2} - R \end{pmatrix},$$

cf. [31, Thm. 23]. The assertion follows from [2, Théo. 1.1]. \square

Definition 4.2. *A permutation O on V is called a node automorphism of $G = (V, E, \mu, \nu)$ if for all $v, w \in V$*

- $\nu(Ov) = \nu(v)$ and
- the entries of \mathcal{A} introduced in (2.1) satisfy $\alpha_{Ov, Ow} = \alpha_{vw}$ (i.e., $(Ov, Ow) \in E$ or $(Ow, Ov) \in E$ if and only if $(v, w) \in E$ or $(w, v) \in E$, and in this case $\mu(Ov, Ow) = \mu(v, w)$).

We denote by $\text{Aut}(G)$ the group of all node automorphisms of V .

In the unweighted graph, the above definition reduces to the usual one: node automorphisms are permutations on V that preserve the adjacency relation. We emphasize that this definition does not depend on the orientation of G .

Observe that each node automorphism O induces a permutation O_L on E by

$$O_L e = (Ov, Ow) \quad \text{if } e = (v, w).$$

Thus, each edge permutation O defines a mapping on \mathbb{R}^V and a mapping on \mathbb{R}^E defined as the Nemitskii operators associated with O and O_L , respectively, i.e.,

$$f \mapsto f(O \cdot) \quad \text{and} \quad u \mapsto u(O_L \cdot).$$

With a slight abuse of notation, we will not distinguish between the node/edge permutations and their associated Nemitskii operators.

Theorem 4.3. *Let $p \in (1, \infty)$. If $O \in \text{Aut}(\mathbf{G})$ and $a \equiv \mu$ as well as $d \equiv \nu$ (i.e., $\mathcal{L}_p = \Delta_p$ as in (2.2)), then*

$$e^{-t\Delta_p} O^k = O^k e^{-t\Delta_p} \quad \text{for all } t \geq 0 \text{ and } k \in \mathbb{N}.$$

If \mathbf{G} is unweighted, this result is clear for $p = 2$, since then $\Delta = D - A$ for a diagonal matrix D .

Proof. Clearly, it suffices to prove the claimed commutation relation for $k = 1$. This can be checked owing to Lemma 4.1, observing that setting $\Sigma := O$ one has

$$L = \frac{1}{2} I_{H_1} \quad \text{and} \quad R = \frac{1}{2} I_{H_2}.$$

Indeed,

$$\begin{aligned} & \mathcal{E}_p \left(\frac{f + O^* g}{2} \right) + \mathcal{E}_p \left(\frac{Of + g}{2} \right) \\ &= \frac{1}{p} \sum_{\mathbf{e} \in E} a(\mathbf{e}) \left| \frac{(\mathcal{I}^T f)(\mathbf{e}) + (\mathcal{I}^T O^* g)(\mathbf{e})}{2} \right|^p + \frac{1}{p} \sum_{\mathbf{e} \in E} a(\mathbf{e}) \left| \frac{(\mathcal{I}^T g)(\mathbf{e}) + (\mathcal{I}^T Of)(\mathbf{e})}{2} \right|^p. \end{aligned}$$

By a change of variables and using the fact that a is constant along induced orbits we obtain

$$\begin{aligned} & \mathcal{E}_p \left(\frac{f + O^* g}{2} \right) + \mathcal{E}_p \left(\frac{Of + g}{2} \right) \\ &= \frac{1}{p} \sum_{O_L \mathbf{e} \in E} a(\mathbf{e}) \left| \frac{(\mathcal{I}^T Of)(\mathbf{e}) + (\mathcal{I}^T g)(\mathbf{e})}{2} \right|^p + \frac{1}{p} \sum_{\mathbf{e} \in E} a(\mathbf{e}) \left| \frac{(\mathcal{I}^T Of)(\mathbf{e}) + (\mathcal{I}^T g)(\mathbf{e})}{2} \right|^p \\ &= \frac{1}{p} \frac{1}{2^{p-1}} \sum_{\mathbf{e} \in E} a(\mathbf{e}) |(\mathcal{I}^T Of)(\mathbf{e}) + (\mathcal{I}^T g)(\mathbf{e})|^p \\ &\leq \frac{1}{p} \left(\sum_{\mathbf{e} \in E} a(\mathbf{e}) \left(|(\mathcal{I}^T Of)(\mathbf{e})|^p + |(\mathcal{I}^T g)(\mathbf{e})|^p \right) \right) \\ &= \frac{1}{p} \left(\sum_{\mathbf{e} \in E} a(\mathbf{e}) \left(|(\mathcal{I}^T f)(\mathbf{e})|^p + |(\mathcal{I}^T g)(\mathbf{e})|^p \right) \right) \\ &= \mathcal{E}_p(f) + \mathcal{E}_p(g), \end{aligned}$$

as we wanted to prove. □

Remark 4.4. (1) Let \mathbf{G} have finite surface. Averaging a function u over all nodes (i.e., shorting all nodes, in the point of view of Remark 2.10) one obtains a system which is trivially left invariant under the evolution of the p -heat equation, since the associated projection P satisfies of course $\mathcal{E}_p(\mathbb{1}) \equiv 0$. This is independent on the automorphism group of \mathbf{G} , see Remark 4.10.(2) below.

(2) On the other hand, just shorting two arbitrary nodes is not sufficient to obtain an invariant subsystem: this can be easily seen by taking a path of length 3 with $\mu \equiv 1$ and considering the projection P of $\ell^2(V) \equiv \mathbb{R}^4$ onto the space

$$\{u : \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \rightarrow \mathbb{R} : u(\mathbf{v}_2) = u(\mathbf{v}_3)\}.$$

Then, for $u(v_n) := n$ one has

$$2 \left(\frac{3}{2} \right)^p = p\mathcal{E}_p(Pu) \not\leq p\mathcal{E}_p(u) = 3, \quad p > 1,$$

which by [2, Théo. 1.1] shows that said subspace is not invariant under $e^{-t\Delta_p}$.

Clearly, each subgroup of $\text{Aut}(\mathbf{G})$ defines a partition of \mathbf{V} into equivalence classes with respect to their orbits. We denote by $[v]$ such orbits and by $||[v]||_d$ their lengths with respect to d , i.e.,

$$||[v]||_d := \sum_{w \in [v]} d(w).$$

Theorem 4.5. *Let $p \in (1, \infty)$. Let Γ be a subgroup of $\text{Aut}(\mathbf{G})$ with orbits of finite length with respect to ν . Let $a \equiv \mu$ as well as $d \equiv \nu$ (i.e., $\mathcal{L}_p = \Delta_p$ as in (2.2)) and consider the orbitwise averaging operator P defined by*

$$Pf(v) := \frac{1}{||[v]||_\nu} \sum_{w \in [v]} f(w)\nu(w), \quad f \in \ell_\nu^2(\mathbf{V}), v \in \mathbf{V}.$$

Then the following assertions hold.

- (1) The range of P is left invariant under $(e^{-t\Delta_p})_{t \geq 0}$.
- (2) The null space of P is left invariant under $(e^{-t\Delta_p})_{t \geq 0}$ if $p = 2$.

Example 4.6. 1) A typical and relevant case is that of an infinite radial trees \mathbf{T} , i.e., infinite rooted trees such that any two nodes with the same distance n from the root have the same number γ_n of children – for the sake of simplicity, say, \mathbf{T} is binary, i.e., $\gamma_n = 2$ for all $n \in \mathbb{N}$. In this case, the automorphism group of \mathbf{T} is abstractly isomorphic to a semidirect product of permutation groups and any orbitwise projection acts by simply averaging ℓ^2 -functions supported in an infinite binary subtree \mathbf{T}_0 of \mathbf{T} over all nodes at the same distance from the root of \mathbf{T}_0 . Then, Theorem 4.5 shows that radial initial data give rise to radial solutions. This is already known in the linear case $p = 2$, both in the case of discrete and metric graphs ([36, 40]). The case of graphs that are not trees has been studied in [18].

2) The range of P with respect to the subgroup Γ is isomorphic to a new “quotient graph” \mathbf{G}/Γ obtained identifying all the nodes belonging to the same orbit; in the case of radial trees, if Γ is the full automorphism group, then \mathbf{G}/Γ is a semi-infinite path, i.e., \mathbb{N} .

Instead of proving the above theorem directly, we will deduce it as a corollary of a more general result.

Definition 4.7. Let I be a (possibly infinite) set. A partition

$$\mathbf{V} = \dot{\bigcup}_{i \in I} \mathbf{V}_i$$

of the node set \mathbf{V} of \mathbf{G} is said to be almost equitable with cells $(\mathbf{V}_i)_{i \in I}$ if for all $i, j \in I$, $i \neq j$,

$$(4.1) \quad \text{there are numbers } c_{ij} > 0 \text{ s.t. } \sum_{w \in \mathbf{V}_j} \mu(v, w) = c_{ij}\nu(v) \quad \text{for all } v \in \mathbf{V}_i.$$

We write

$$|\mathbf{V}_i|_\nu := \sum_{v \in \mathbf{V}_i} \nu(v), \quad i \in I.$$

Remark 4.8. (1) Clearly, each node partition of a graph canonically induces an edge partition: Simply take edge cells \mathbf{E}_{ij} consisting of all edges with initial endpoint in \mathbf{V}_i and terminal one in \mathbf{V}_j (orientation does matter! – that is, $\mathbf{E}_{ij} \neq \mathbf{E}_{ji}$). Moreover, we let $[e] := \mathbf{E}_{ij}$ if $e \in \mathbf{E}_{ij}$ and write

$$(4.2) \quad |\mathbf{E}_{ij}|_a := \sum_{e \in \mathbf{E}_{ij}} a(e), \quad i, j \in I.$$

(2) In particular, if $(V_i)_{i \in I}$ is an almost equitable partition, then $|E_{ij}|_\mu = c_{ij}|V|_\nu$ for all $i, j \in I$ s.t. $i \neq j$.

(3) If the condition (4.1) holds for all i, j (and not only for $i \neq j$), then the partition is called equitable. Our definition is a generalization of the classical one for unweighted graphs ($\mu \equiv 1, \nu \equiv 1$), see e.g. [20, 30] for equitable partitions and [5, § 2.3] for almost equitable ones. In the unweighted case, existence of an equitable node partition amounts to saying that each node in V_i is adjacent to exactly c_{ij} nodes in V_j .

Then, the quotient graph is the directed, weighted multigraph with node set of cardinality $|I|$ (the i -th node w_i corresponding to the cell V_i) such that (w_i, w_j) is an edge (and if so, with weight c_{ij}) if and only if $c_{ij} \neq 0$. A simple but useful result in the linear, finite, unweighted case is that the spectrum the discrete Laplacian of G contains that of the discrete Laplacian of its quotient graph, see e.g. [30, Thm. 2.3]. In certain special cases the spectra even agree (not counting multiplicity, of course), cf. [10, Thm. 7.8].

Theorem 4.9. *Let $p \in (1, \infty)$. Let G have an equitable partition associated with cells $(V_i)_{i \in I}$ s.t. $|V_i|_d < \infty$ and $|E_{ij}|_a < \infty$ for all $i, j \in I$. Let $a \equiv \mu$ as well as $d \equiv \nu$ (i.e., $\mathcal{L}_p = \Delta_p$ as in (2.2)). Assume that*

$$\text{there exist } a_{ij} > 0 \text{ s.t. } a(e) = a_{ij} \quad \text{for all } e \in E_{ij}, \text{ and all } i, j \in I, i \neq j.$$

Consider the cellwise averaging operator P defined by

$$Pf(v) := \frac{1}{|V_i|_d} \sum_{w \in V_i} f(w) d(w), \quad f \in \ell_d^2(V), v \in V_i.$$

Then the following assertions hold.

- (1) The range of P is left invariant under $(e^{-t\Delta_p})_{t \geq 0}$.
- (2) The null space of P is left invariant under $(e^{-t\Delta_p})_{t \geq 0}$ if $p = 2$.

This can be seen as a nonlinear counterpart of a well-known result of algebraic graph theory, see e.g. [21, Thm. 9.3.3], or a discrete counterpart of the settings in [8, 40] (where different terminologies are used).

Proof. By [2, Théo. 1.1], the range of P is invariant under $(e^{-t\mathcal{L}_p})_{t \geq 0}$ if and only if

$$(4.3) \quad \|\mathcal{I}^T Pf\|_{\ell_a^p}^p \leq \|\mathcal{I}^T f\|_{\ell_a^p}^p \quad \text{for all } f \in w_{a,d}^{1,p,2}(v),$$

whereas its null space is invariant if and only if

$$(4.4) \quad \|\mathcal{I}^T(f - Pf)\|_{\ell_a^p}^p \leq \|\mathcal{I}^T f\|_{\ell_a^p}^p \quad \text{for all } f \in w_{a,d}^{1,p,2}(v).$$

First of all, define the averaging operator

$$\tilde{P}u(e) := \frac{1}{|E_{ij}|_a} \sum_{f \in E_{ij}} u(f) a(f), \quad u \in \ell_\mu^2(E), e \in E \text{ s.t. } e \in E_{ij},$$

and observe that for all $f \in \ell_d^2(V)$, all $e \in E$, and all $i, j \in I$

- $\mathcal{I}^T Pf(e) = 0$ if $e \in E_{ii}$, since $Pf(v) = Pf(w)$ for all $v, w \in V_i$,

- while if $\mathbf{e} \in \mathbf{E}_{ij}$ with $i \neq j$, and hence $\mathbf{e}_0 \in \mathbf{V}_i$ and $\mathbf{e}_1 \in \mathbf{V}_j$, then

$$\begin{aligned}
|\mathcal{I}^T P f(\mathbf{e})| &= |P f(\mathbf{e}_0) - P f(\mathbf{e}_1)| \\
&= \left| \frac{1}{|\mathbf{V}_i|_d} \sum_{\mathbf{v} \in \mathbf{V}_i} f(\mathbf{v}) d(\mathbf{v}) - \frac{1}{|\mathbf{V}_j|_d} \sum_{\mathbf{w} \in \mathbf{V}_j} f(\mathbf{w}) d(\mathbf{w}) \right| \\
&= \left| \left(f \middle| d \left(\frac{\mathbb{1}_{\mathbf{V}_i}}{|\mathbf{V}_i|_d} - \frac{\mathbb{1}_{\mathbf{V}_j}}{|\mathbf{V}_j|_d} \right) \right)_{\ell^2(\mathbf{V})} \right| \\
&\stackrel{(*)}{=} \left| \left(f \middle| \mathcal{I} \frac{a \mathbb{1}_{\mathbf{E}_{ij}}}{|\mathbf{E}_{ij}|_a} \right)_{\ell^2(\mathbf{V})} \right| \\
&= \left| \frac{1}{|\mathbf{E}_{ij}|_a} \sum_{\mathbf{f} \in \mathbf{E}_{ij}} (\mathcal{I}^T f)(\mathbf{f}) a(\mathbf{f}) \right| \\
&= |\tilde{P} \mathcal{I}^T f(\mathbf{e})|,
\end{aligned}$$

where $(*)$ follows from the definition of equitable partition and Remark 4.8.(2).

(2) This shows that, for $p = 2$, we are actually looking for properties of the variance and the expected value of the random variable $X := \mathcal{I}^T f$: hence it suffices to apply the formula $\text{Var}(X) + (E[X])^2 = E|X|^2$ to in order to check that the claimed inequalities hold for $p = 2$.

(1) In order to prove (4.3) for general $p > 1$, we proceed as follows. One has

$$\begin{aligned}
\sum_{\mathbf{e} \in \mathbf{E}} a(\mathbf{e}) |\mathcal{I}^T P f(\mathbf{e})|^p &= \sum_{\substack{i,j \in I \\ i \neq j}} \sum_{\mathbf{e} \in \mathbf{E}_{ij}} a_{ij} |\mathcal{I}^T P f(\mathbf{e})|^p + \sum_{i \in I} \sum_{\mathbf{e} \in \mathbf{E}_{ii}} a(\mathbf{e}) |\mathcal{I}^T P f(\mathbf{e})|^p \\
&= \sum_{\substack{i,j \in I \\ i \neq j}} a_{ij} \sum_{\mathbf{e} \in \mathbf{E}_{ij}} |\tilde{P} \mathcal{I}^T f(\mathbf{e})|^p \\
&= \sum_{\substack{i,j \in I \\ i \neq j}} a_{ij} \sum_{\mathbf{e} \in \mathbf{E}_{ij}} \left| \frac{1}{|\mathbf{E}_{ij}|_a} \sum_{\mathbf{f} \in \mathbf{E}_{ij}} \mathcal{I}^T f(\mathbf{f}) a_{ij} \right|^p \\
&\leq \sum_{\substack{i,j \in I \\ i \neq j}} a_{ij} \sum_{\mathbf{e} \in \mathbf{E}_{ij}} \frac{a_{ij}}{|\mathbf{E}_{ij}|_a} \sum_{\mathbf{f} \in \mathbf{E}_{ij}} |\mathcal{I}^T f(\mathbf{f})|^p.
\end{aligned}$$

where the last step follows from Jensen's inequality. In view of (4.2), we obtain

$$\sum_{\mathbf{e} \in \mathbf{E}} a(\mathbf{e}) |\mathcal{I}^T P f(\mathbf{e})|^p \leq \sum_{\substack{i,j \in I \\ i \neq j}} a_{ij} \sum_{\mathbf{f} \in \mathbf{E}_{ij}} |\mathcal{I}^T f(\mathbf{f})|^p \leq \sum_{\mathbf{e} \in \mathbf{E}} a(\mathbf{e}) |\mathcal{I}^T f(\mathbf{e})|^p.$$

This shows that (4.3) holds and completes the proof. \square

Remark 4.10. (1) The second assertion in Theorem 4.5 is optimal, in the sense that there exist graphs on which the null space of P is not left invariant under $e^{-t\Delta_p}$ for any $p \neq 2$. An example for which (4.4) fails to hold for any $p \neq 2$ is given as follows: Take \mathbf{G} to be a path of length 3, with weights $\mu \equiv 1$ and $\nu \equiv 1$. Consider the trivial equitable partition $([\mathbf{v}] := \{\mathbf{v}\} \text{ for all nodes})$ and the function

$$f \equiv \left(1, \frac{1}{2}, 0, x \right) \in \mathbb{R}^4,$$

for $x \in \mathbb{R}$ to be fitted later. Then

$$\mathcal{I}^T f \equiv \left(-\frac{1}{2}, -\frac{1}{2}, x \right),$$

and choosing values of x slightly larger than 1 for $p > 2$, and slightly smaller than 1 for $p < 2$ yields the sought-after counterexample, by elementary calculus arguments.

(2) We have already observed in Remark 4.4.(1) that averaging over all nodes of a graph with finite surface yields a projection whose range is invariant under $(e^{-t\mathcal{L}_p})_{t \geq 0}$ for any p . This is in fact a special case of Theorem 4.9, since each graph has the (trivial) almost equitable partition given by $\mathbf{V}_1 \equiv \mathbf{V}$.

(3) Theorem 4.9 is strictly more general than Theorem 4.5, since the orbit partition w.r.t. any subgroup Γ of $\text{Aut}(\mathbf{G})$ yields an equitable partition of \mathbf{V} , but the converse is generally false (cf. [20]).

(4) The (non-equitable) partitioning of \mathbf{V} into $\text{Fix}(\Gamma) := \{v \in \mathbf{V} : Ov = v \ \forall O \in \Gamma\}$ and its complement is in general not respected by $(e^{-t\mathcal{L}_p})_{t \geq 0}$ for any p , as the simple following example shows: Take \mathbf{G}_m as an unweighted m -star with a path of length 2 attached to the center, and consider the function f defined by $f(v) = 0$ on each node v of the star (including the center) and $f(v) = 1$ on both the remaining nodes.

5. APPENDIX: FURTHER DISCRETE OPERATORS

In this section we comment on several different possible extensions of the theory introduced above.

5.1. Generalized Laplacians. Fiedler, Colin de Verdière and others have introduced and studied several variations on the classic discrete Laplacian. The following one, which goes back to [12], is the most usual one: Given an simple, finite, connected graph with node set \mathbf{V} , any $|\mathbf{V}| \times |\mathbf{V}|$ -matrix whose off-diagonal \mathbf{v} - \mathbf{w} -entry is

- = 0 if and only there is no edge between \mathbf{v} and \mathbf{w} and
- < 0 otherwise

is called a *generalized Laplacian*. Besides the discrete Laplacian, also $-\mathcal{A}$ (where \mathcal{A} is the adjacency matrix) and Chung's normalized Laplacian ([10, § 1.4])

$$\mathcal{P} := D^{-\frac{1}{2}} \Delta D^{-\frac{1}{2}}$$

(where D is the diagonal matrix whose i - i -entry is the degree of \mathbf{v}_i) are clearly generalized Laplacians. Unlike the adjacency matrix, Chung's \mathcal{P} is also positive definite, since

$$\mathcal{P} = (D^{-\frac{1}{2}} \mathcal{I})(D^{-\frac{1}{2}} \mathcal{I})^T.$$

The *signless Laplacian*

$$\mathcal{Q} := D + A$$

has been introduced in [14]. Its study has gained much momentum in the last decade, also owing to thorough investigations by Cvetković and others, cf. [13]. (Clearly, $-\mathcal{Q}$ is a generalized Laplacian, too). One possible reason for this popularity is the richness of interesting graph theoretical properties of \mathcal{Q} (e.g., 0 is always an eigenvalue whose multiplicity is the number of bipartite components); another one is its nice variational structure. In particular, one easily sees that

$$\mathcal{Q} := \mathcal{J} \mathcal{J}^T,$$

where \mathcal{J} is the incidence matrix of the undirected graph underlying \mathbf{G} , i.e., $\mathcal{J} := \mathcal{I}^+ + \mathcal{I}^-$. One may further generalize this class of Laplacians by taking $\sigma \in \ell_V^\infty(\mathbf{V})$ and letting

$$(\mathcal{J}_\sigma)_{\mathbf{ve}} := \mathcal{I}_{\mathbf{ve}}^+ + \sigma(\mathbf{v}) \mathcal{I}_{\mathbf{ve}}^-, \quad \mathbf{v} \in \mathbf{V}, \mathbf{e} \in \mathbf{E}.$$

We may then introduce

$$\mathcal{Q}^\sigma := \mathcal{J}_\sigma \mathcal{J}_\sigma^T,$$

which is still a positive definite matrix and also a generalized Laplacian (each of whose non-vanishing off-diagonal entries is one of the $\sigma_{\mathbf{v}}$); along with its normalized version

$$\mathcal{P}^\sigma := (D^{-\frac{1}{2}} \mathcal{J}_\sigma)(D^{-\frac{1}{2}} \mathcal{J}_\sigma)^T.$$

(This generalization process could continue by allowing for more general matrices σ , but this would necessarily destroy locality of \mathcal{Q}^σ). It is then natural to consider a *signless p -Laplacian* by

$$\mathcal{Q}_p f := \mathcal{J}(|\mathcal{J}^T f|^{p-1} \text{sign}(\mathcal{J}^T f)),$$

or, more generally,

$$\mathcal{Q}_p^\sigma f := \mathcal{J}_\sigma(|\mathcal{J}_\sigma^T f|^{p-1} \text{sign}(\mathcal{J}_\sigma^T f)).$$

Likewise, one can introduce the normalized operator

$$\mathcal{P}_p^\sigma f := D^{-\frac{1}{2}} \mathcal{J}_\sigma(|\mathcal{J}_\sigma^T D^{-\frac{1}{2}} f|^{p-1} \text{sign}(\mathcal{J}_\sigma^T D^{-\frac{1}{2}} f)).$$

We are aware of only a few previous investigations on the signless p -Laplacian in the literature, including [3, 23], where it is proved that the sets of eigenvalues of \mathcal{Q}_p and Δ_p agree, provided that \mathbf{G} is bipartite.

Remark 5.1. *In the linear case $p = 2$, the parabolic theory of the signless Laplacian on uniformly locally finite graphs is not overly interesting. This is due to the fact that $\mathcal{Q} + \Delta = 2D$ (we are omitting the index 2 of Δ_2, \mathcal{Q}_2). All these three operators are bounded. Although D and \mathcal{A} (and therefore D and Δ, \mathcal{Q}) do not commute, so that in general*

$$e^{t\mathcal{Q}} \neq e^{-t\Delta} e^{2tD}, \quad t \in \mathbb{R},$$

Lie's product formula still holds and yields

$$e^{t\mathcal{Q}} = \lim_{n \rightarrow \infty} \left(e^{-\frac{t}{n}\Delta} e^{2\frac{t}{n}D} \right)^n, \quad t \in \mathbb{R}.$$

Alternatively, we can recover another formula for $(e^{t\mathcal{Q}})_{t \geq 0}$ based on the Dyson–Phillips formula. Since D is a positive diagonal matrix, knowing qualitative information on $(e^{-t\Delta})_{t \in \mathbb{R}}$ promptly yields quite complete information on $(e^{t\mathcal{Q}})_{t \in \mathbb{R}}$, and vice versa. For example, since $(e^{-t\Delta})_{t \geq 0}$ is a positive semigroup, so is $(e^{t\mathcal{Q}})_{t \geq 0}$.

Things are different in the general nonlinear case, since it is not clear in which sense \mathcal{Q}_p can be seen as a perturbation of Δ_p (unless p is an even natural number, in which case Δ_p and \mathcal{Q}_p can be compared using the binomial formula).

The following can be proved like in the case of Δ_p .

Proposition 5.2. *Let $p \in (1, \infty)$ and consider the functional $\mathcal{F}_p : \ell_d^2(\mathbf{V}) \rightarrow [0, \infty]$ defined by*

$$\mathcal{F}_p^\sigma : u \mapsto \frac{1}{p} \|\mathcal{J}_\sigma f(\mathbf{e})\|_{\ell_a^p(\mathbf{E})}^p = \frac{1}{p} \sum_{\mathbf{e} \in \mathbf{E}} |f(\mathbf{e}_0) + \sigma(\mathbf{e}_1)f(\mathbf{e}_1)|^p.$$

Then the following assertions hold.

(1) *The effective domain*

$$z_{a,d}^{1,p,2}(\mathbf{V}) := \{f \in \ell_d^2(\mathbf{V}) : \mathcal{J}_\sigma^T f \in \ell_a^p(\mathbf{E})\}.$$

of \mathcal{F}_p is a Banach space with respect to the norm defined by

$$\|f\|_{z_{a,d}^{1,p,2}} := \sqrt{\|f\|_{\ell_d^2}^2 + \|\mathcal{J}_\sigma^T f\|_{\ell_a^p}^p}.$$

For all $p \in [1, \infty]$, $z_{a,d}^{1,p,2}(\mathbf{V})$ is continuously and densely embedded into $\ell_d^2(\mathbf{V})$. If moreover $p \in [1, \infty)$, then $z_{a,d}^{1,p,2}(\mathbf{V})$ is separable. If $p \in (1, \infty)$, then $z_{a,d}^{1,p,2}(\mathbf{V})$ is uniformly convex (and hence reflexive).

- (2) *The functional \mathcal{F}_p^σ is proper and convex. It is continuously (Fréchet) differentiable as a functional on $z_{a,d}^{1,p,2}(\mathbf{V})$, while it is lower semicontinuous as a functional on $\ell_d^2(\mathbf{V})$.*
- (3) *The subdifferential \mathcal{Q}_p^σ of \mathcal{F}_p^σ generates a strongly continuous semigroup of nonlinear contractions on $\ell_d^2(\mathbf{V})$.*
- (4) *If $\sigma \equiv 1$, then the set of eigenvectors of \mathcal{Q}_p^σ for the eigenvalue 0 form a subspace of $\ell_d^2(\mathbf{V})$ whose dimension is the number of bipartite components of \mathbf{G} with finite surface.*

Observe that unless $\sigma = \pm 1$, \mathcal{F}_p^σ and hence its subdifferential \mathcal{Q}_p^σ do depend on the orientation of \mathbf{G} .

Proof. Because

$$\mathcal{F}_p^\sigma = \frac{1}{p} \|\cdot\|_{\ell_a^p}^p \circ \mathcal{J}_\sigma^T,$$

and because \mathcal{J}_σ is a bounded linear operator from $z_{a,d}^{1,p,2}(\mathbf{V})$ to $\ell_a^p(\mathbf{E})$, all the proofs are analogous to those for the corresponding assertions about \mathcal{F}_p^σ and Δ_p . The only minor change is needed in the proof of (4), where the dimension formula for the null space of \mathcal{I}^T has to be replaced by a dimension formula for the null space of \mathcal{J}_1^T , cf. [42]. \square

5.2. The $p(\mathbf{e})$ -Laplacian. We could also consider the equivalent of the so-called $p(x)$ -Laplacian: in our discrete setting the associated energy functional is given by

$$\mathcal{E}_p(f) := \sum_{\mathbf{e} \in \mathbf{E}} \frac{a(\mathbf{e})}{p(\mathbf{e})} |f(\mathbf{e}_0) - f(\mathbf{e}_1)|^{p(\mathbf{e})}$$

under suitable assumptions on $p \in \mathbb{R}^{\mathbf{E}}$. In this case it would not in general be the subdifferential of \mathcal{E}_p , but rather that of its convex, lower semicontinuous relaxation that generates a semigroup.

5.3. Discrete Schrödinger operators. Another possible direction of generalization consists of considering an energy functional with additional terms defined by

$$\mathcal{E}_p(f) := \frac{1}{p} \sum_{\mathbf{e} \in \mathbf{E}} a(\mathbf{e}) |f(\mathbf{e}_0) - f(\mathbf{e}_1)|^p + \frac{1}{2} \sum_{\mathbf{v} \in \mathbf{V}} b(\mathbf{v}) |f(\mathbf{v})|^2.$$

for some b asymptotically comparable with d . Due to their interpretation in (linear) potential theory, the terms in the second sum are sometimes referred to as *killing terms*. The subdifferential of \mathcal{E}_p is in fact the discrete analogue of a Schrödinger operator with scalar potential b . We refer to [27] and references therein for a comprehensive theory in the linear case.

5.4. Discrete operators with boundary conditions at infinity. One can introduce the space $\dot{w}_{a,d}^{1,p,2}(\mathbf{V})$ defined as the closure of $c_{00}(\mathbf{V})$ with respect to the norm of $w_{a,d}^{1,p,2}(\mathbf{V})$. Whenever \mathbf{G} is finite, $\dot{w}_{a,d}^{1,p,2}(\mathbf{V})$ always agrees with $w_{a,d}^{1,p,2}(\mathbf{V})$ (in fact, both agree with $\ell_d^2(\mathbf{V})$), but for infinite graphs things are less obvious. A characterization of the quotient space

$$w^{1,2,2}(\mathbf{G}) / \dot{w}^{1,2,2}(\mathbf{G}).$$

is a classic topic of potential theory and in special cases it leads to the introduction of the so-called Martin boundary (see e.g. [35] for its connection to the notion of space of ends of an infinite graph), but in the general case we are only aware of a subtle discussion of parabolic and graph theoretical properties implying non-triviality of $w^{1,2,2}(\mathbf{G}) / \dot{w}^{1,2,2}(\mathbf{G})$ performed in [24, §§ 4–5]. While the potential theory of discrete boundary value problems seems to be confined to the linear case therein, one may introduce also in our case the quotient Banach space

$$b_{a,d}^p(\partial \mathbf{G}) := w_{a,d}^{1,p,2}(\mathbf{G}) / \dot{w}_{a,d}^{1,p,2}(\mathbf{G}).$$

If Tr denotes the canonical surjection of $w_{a,d}^{1,p,2}(\mathbf{G})$ onto $b_{a,d}^p(\partial \mathbf{G})$, one may consider the general functional

$$\tilde{\mathcal{E}}_{p,q}(f) := \frac{1}{p} \|\mathcal{I}^T f\|_{\ell_a^p}^p + \frac{1}{q} \|\text{Tr } f\|_{b_{a,d}^p(\partial \mathbf{V})}^q.$$

Then, it is easy to check that $\mathcal{E}_{m,q}$ is proper, lower semicontinuous and convex and the associated subdifferential is a discrete p -Laplacian with discrete Robin-type boundary conditions.

5.5. The porous medium equation. We briefly discuss the interplay between porous medium equation and p -heat equation on graphs. For the sake of simplicity, we only consider the unweighted case, i.e., $\mu = a \equiv 1$ and $\nu = d \equiv 1$. The continuous porous medium equation can be studied in the context of the theory of subdifferentials, see e.g. [38, Examples III.6.C and IV.6.B]; but it is known that *in the continuous, 1-dimensional case* if ψ solves the porous medium equation

$$\dot{\psi} = (|\psi|^{\pi-1}\psi)_{xx},$$

then there is ϕ such that ψ is its pressure (i.e., $\phi_x = \psi$) and it satisfies the p -heat equation

$$\dot{\phi} = (|\phi_x|^{p-2}\phi_x)_x$$

for $p : f = \pi + 1$. While the proof in is based on the theory of exact differential forms (see e.g. [43, § 3.4.3]), and it is not clear whether it has a pendant in our context, this argument roughly suggests that the correct space to discuss the porous medium equation is $\ell^2(\mathbf{E})$, which we can look at as the node space of the line graph \mathbf{G}_L of \mathbf{G} and consider as a *pressure space*. We can prove a similar result in a special class of oriented bipartite graphs.

Observe that bipartition induces a natural orientation: if $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$, then assume all edges to have initial endpoint in \mathbf{V}_1 (and hence terminal endpoint in \mathbf{V}_2). We recall that a bipartite graph ($\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$) is called *semiregular* if all nodes in \mathbf{V}_i have same degree d_i , $i = 1, 2$.

Proposition 5.3. *Let \mathbf{G} be uniformly locally finite and $p \geq 2$. Let $T > 0$ and $f \in L^2(0, T; \ell^2(\mathbf{V}))$. If $\phi \in H^1(0, T; \ell^2(\mathbf{V}))$ satisfies*

$$\dot{\phi}(t, \mathbf{v}) = -\mathcal{I}(|\mathcal{I}^T \phi|^{p-2} \mathcal{I}^T \phi)(t, \mathbf{v}) + f(t), \quad t > 0, \mathbf{v} \in \mathbf{V},$$

then $\psi := \mathcal{I}^T \phi \in H^1(0, T; \ell^2(\mathbf{E}))$ satisfies

$$\dot{\psi}(t, \mathbf{e}) = -\mathcal{I}^T \mathcal{I}(|\psi|^{\pi-1} \psi)(t, \mathbf{e}) + \mathcal{I}^T f(t), \quad t > 0, \mathbf{e} \in \mathbf{E},$$

for $p = \pi + 1$. Furthermore, ψ satisfies

$$(5.1) \quad \dot{\psi}(t, \mathbf{e}) = \Delta_{\mathbf{G}_L}(|\psi|^{\pi-1} \psi)(t, \mathbf{e}) - (r + s)(|\psi|^{\pi-1} \psi)(t, \mathbf{e}) + \mathcal{I}^T f(t), \quad t > 0, \mathbf{e} \in \mathbf{E},$$

if additionally \mathbf{G} is (r, s) -semiregular bipartite and is oriented accordingly.

Observe that (5.1) is a *forward* porous medium-type equation (with potential) on the line graph \mathbf{G}_L of \mathbf{G} . We have denoted by $\Delta_{\mathbf{G}_L}$ the (linear) discrete Laplacian on \mathbf{G}_L . We stress that *the latter result is not independent of orientation of \mathbf{G} .*

Proof. It is a direct consequence of Lemma 2.7 that $\dot{\psi} = \mathcal{I}^T \dot{\phi}$ if \mathbf{G} is uniformly locally finite.

Let now \mathbf{G} be bipartite and oriented accordingly. The second assertion follows recalling that

$$\mathcal{I}^T \mathcal{I} = (r + s)\text{Id} - \Delta_{\mathbf{G}_L}$$

whenever \mathbf{G} is (r, s) -semiregular, see e.g. [30, Proof of Thm. 3.9]. □

Consequently, by Theorem 3.7 and Corollary 2.9 the following holds, since an infinite graph has infinite surface if $\nu \equiv 1$.

Corollary 5.4. *Let $T > 0$. If in particular \mathbf{G} is a forest each of whose connected components is an infinite, uniformly locally finite tree, then for all $g \in L^2(0, T; \ell^2(\mathbf{E}))$ and all $\psi_0 \in \ell^2(\mathbf{E})$ there exists a unique solution $\psi \in H^1(0, T; \ell^2(\mathbf{E}))$ to*

$$(PME) \quad \begin{cases} \dot{\psi}(t, \mathbf{v}) &= \mathcal{I}^T \mathcal{I}(|\psi|^{\pi-1} \psi)(t, \mathbf{v}) + g(t), & t \geq 0, \mathbf{v} \in \mathbf{V}, \\ \psi(0, \mathbf{v}) &= \psi_0(\mathbf{v}), & \mathbf{v} \in \mathbf{V}. \end{cases}$$

REFERENCES

- [1] S. Amghibech. Bounds for the largest p -Laplacian eigenvalue for graphs. *Disc. Math.*, 306:2762–2771, 2006.
- [2] L. Barthélemy. Invariance d'un convexe fermé par un semi-groupe associé à une forme non-linéaire. *Abstr. Appl. Analysis*, 1:237–262, 1996.
- [3] T. Bıykoğlu, M. Hellmuth, and J. Leydold. Largest eigenvalues of the discrete p -Laplacian of trees with degree sequences. *Electron. J. Linear Algebra*, 18:202–210, 2009.
- [4] H. Brézis. *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*. North-Holland, Amsterdam, 1973.
- [5] A. Brouwer and W. Haemers. *Spectra of Graphs*. Springer-Verlag, Berlin, 2012.
- [6] T. Bühler and M. Hein. Spectral clustering based on the graph p -Laplacian. In *Proc. 26th Annual Int. Conf. Mach. Learning*, pages 81–88, New York, 2009. ACM.
- [7] S. Cardanobile. The L^2 -strong maximum principle on arbitrary countable networks. *Lin. Algebra Appl.*, 435:1315–1325, 2011.
- [8] S. Cardanobile, D. Mugnolo, and R. Nittka. Well-posedness and symmetries of strongly coupled network equations. *J. Phys. A*, 41:055102, 2008.
- [9] R. Chill and E. Fašangová. *Gradient Systems*. MatFyzPress, Prague, 2010.
- [10] F. Chung. *Spectral Graph Theory*, volume 92 of *Reg. Conf. Series Math.* Amer. Math. Soc., Providence, RI, 1997.
- [11] F. Cipriani and G. Grillo. Nonlinear Markov semigroups, nonlinear Dirichlet forms and applications to minimal surfaces. *J. Reine Ang. Math.*, 562:201–235, 2003.
- [12] Y. Colin de Verdière. Sur un nouvel invariant des graphes et un critère de planarité. *J. Comb. Theory. Ser. B*, 50:11–21, 1990.
- [13] D. Cvetković, P. Rowlinson, and S. Simić. Signless Laplacians of finite graphs. *Lin. Algebra Appl.*, 423:155–171, 2007.
- [14] M. Desai and V. Rao. A characterization of the smallest eigenvalue of a graph. *J. Graph Theory*, 18:181–194, 1994.
- [15] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 2005.
- [16] P. Doyle and J. Snell. *Random Walks and Electric Networks*, volume 22 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington (DC), 1984.
- [17] P. Drábek. The p -Laplacian – mascot of nonlinear analysis. *Acta Math. Univ. Comenianae*, 76:85–98, 2007.
- [18] D. E. Dutkay and P. E. T. Jorgensen. Spectral theory for discrete laplacians. *Compl. Anal. Oper. Theory*, 4:1–38, 2010.
- [19] A. Elmoataz, O. Lezoray, and S. Boughleux. Nonlocal discrete regularization on weighted graphs: a framework for image and manifold processing. *IEEE Trans. Image Processing*, 17:1047–1060, 2008.
- [20] C. Godsil. Compact graphs and equitable partitions. *Lin. Algebra Appl.*, 255:259–266, 1997.
- [21] C. Godsil and G. Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 2001.
- [22] L. Grady and J. Polimeni. *Discrete Calculus: Applied Analysis on Graphs for Computational Science*. Springer-Verlag, New York, 2010.
- [23] Z. Guang-Jun and Z. Xiao-Dong. The p -Laplacian spectral radius of weighted trees with a degree sequence and a weight set. *Electron. J. Linear Algebra*, 22:267–276, 2011.
- [24] S. Haeseler, M. Keller, D. Lenz, and R. Wojciechowski. Laplacians on infinite graphs: Dirichlet and Neumann boundary conditions. arXiv:1103.3695, 2011.
- [25] I. Holopainen and P. Soardi. A strong Liouville theorem for p -harmonic functions on graphs. *Ann. Acad. Sci. Fen.*, 22:205–226, 1997.
- [26] D. Jiang, J. Chu, D. O'Regan, and R. Agarwal. Positive solutions for continuous and discrete boundary value problems to the one-dimension p -Laplacian. *Math. Inequal. Appl.*, 7:523–534, 2004.

- [27] M. Keller and D. Lenz. Dirichlet forms and stochastic completeness of graphs and subgraphs. *J. Reine Angew. Math.*, (in press).
- [28] G. Kirchhoff. Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird. *Ann. Physik*, 12:497–508, 1847.
- [29] J. Lions. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*. Dunod, Paris, 1969.
- [30] B. Mohar. The Laplacian spectrum of graphs. *Graph theory, combinatorics, and applications*, 2:871–898, 1991.
- [31] J. Neuberger. *Sobolev Gradients and Differential Equations*, volume 1670 of *Lect. Notes Math.* Springer-Verlag, Berlin, 1997.
- [32] R. Nittka. *Elliptic and Parabolic Problems with Robin Boundary Conditions on Lipschitz Domains*. PhD thesis, Universität Ulm, 2011.
- [33] E. Ouhabaz. *Analysis of Heat Equations on Domains*, volume 30 of *Lond. Math. Soc. Monograph Series*. Princeton Univ. Press, Princeton, 2005.
- [34] K. Pankrashkin. Reducible boundary conditions in coupled channels. *J. Phys. A*, 38:8979–8992, 2005.
- [35] M. Picardello and W. Woess. Martin boundaries of random walks: ends of trees and groups. *Trans. Amer. Math. Soc.*, 302:185–205, 1987.
- [36] M. Picardello and W. Woess. A converse to the mean value property on homogeneous trees. *Trans. Amer. Math. Soc.*, 311:209–225, 1989.
- [37] L. Saloff-Coste. Some inequalities for superharmonic functions on graphs. *Potential Analysis*, 6:163–181, 1997.
- [38] R. Showalter. *Monotone Operator in Banach Space and Partial Differential Equations*, volume 49 of *Math. Surveys and Monographs*. Amer. Math. Soc., Providence, RI, 1997.
- [39] P. Soardi. Rough isometries and Dirichlet finite harmonic functions on graphs. *Proc. Amer. Math. Soc.*, 119:1239–1248, 1993.
- [40] M. Solomyak. On the spectrum of the Laplacian on regular metric trees. *Waves Random Media*, 14:155–171, 2004.
- [41] A. Szlam and X. Bresson. Total variation and Cheeger cuts. In *Proc. 27th Annual Int. Conf. Mach. Learning*, pages 1039–1046, New York, 2010. ACM.
- [42] C. Van Nuffelen. On the incidence matrix of a graph. *IEEE Trans. Circuits and Systems*, 23:572–572, 1976.
- [43] J. Vázquez. *The Porous Medium Equation: Mathematical Theory*. Oxford University Press, Oxford, 2007.
- [44] M. Yamasaki. Parabolic and hyperbolic infinite networks. *Hiroshima Math. J.*, 7:135–146, 1977.
- [45] D. Zhou and B. Schölkopf. Regularization on discrete spaces. *Pattern Recognition*, pages 361–368, 2005.

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